Lectures on Singularity formation in the Euler Equations

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Abstract

These lectures were given at the Fields Institute's summer school on Mathematical Hydrodynamics. The purpose of these lectures is to give a quick introduction to some tools that can be used to study the finite-time singularity problem for solutions to the 3d Euler equation.

1 Introduction

These lectures concern the problem of singularity formation for the incompressible Euler equation:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \tag{1.1}$$

$$\mathrm{div}(u) = 0. \tag{1.2}$$

Here, $u : \mathbb{R}^d \times [0, T) \to \mathbb{R}^d$ represents the velocity field of an *incompressible* fluid flowing through all of space. p is the force of internal pressure that acts to keep the constraint (1.2). If we were to disregard (1.2) (i.e. with $p \equiv 0$), we get the multi-dimensional Burgers equation:

$$\partial_t u + u \cdot \nabla u = 0,$$

and introducing $\nabla u = M$, we see that

$$\partial_t M + u \cdot \nabla M = -M^2,$$

from which it is easy to see that ∇u becomes infinite in finite time for most initial data. This finite-time singularity occurs in *all dimensions* for the Burgers equation. A remarkable fact is that the incompressibility constraint (1.2) prevents this type of behavior in dimension $d = 2$. Indeed, introducing

$$\omega := \nabla^{\perp} \cdot u,$$

we see that

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

Thus we see that

$$\omega = \nabla^{\perp} \cdot u \;\text{ is transported by}\, u, \;\text{ while }\; \nabla \cdot u = 0.$$

These two facts together give us the classical

Theorem 1 (Yudovich)**.** *Let u_0 be such that $\omega_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$. Then there is a unique solution to* (1.1)-(1.2) *with*[1] $\omega \in L^1 \cap L^\infty$ *for all* $t \geq 0$.

[1]The correct statement is $\omega \in C(\mathbb{R}; L^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R})$

Natural Question: Is the incompressibility constraint sufficient to prevent a singularity when $d > 2$?

1.1 The vorticity equation in 3d

There is a very important distinction between $d = 2$ and $d = 3$. Upon considering the 3d vorticity:

$$\omega := \nabla \times u.$$

We note that the equation becomes:

$$\partial_t \omega + u \cdot \nabla \omega = \nabla u \, \omega,$$

while the right hand side vanished identically in 2d. The term on the right side is called the bf vortex stretching term and it is what is responsible for growth of ω. Because of this term, the classical well-posedness results that are available on the 3d Euler equation are just local in time:

Theorem 2. *(Lichtenstein, Hölder) Given $\omega_0 \in C^\alpha_c$, there is a $T_* = T_*(\omega_0) > 0$ and a unique solution $\omega \in C^\alpha(\mathbb{R}^3 \times [0, T_*))$.*

Several conditions for being able to continue the solution up to a given T_* were given by many authors over the years. The most famous criteria are due to Beale-Kato-Majda and Constantin-Fefferman-Majda.

1.2 How could a blow-up occur in the vorticity equation?

Upon analyzing the vortex stretching term, in order to capture a blow-up we need to understand several aspects of the mechanism, including:

- (Geometric) **In order for there to be a blow-up, $\nabla u \, \omega$ must be "aligned" with ω.** It is not difficult to construct situations where this does not happen.
- (Nonlocal) **We have to know that there is a mechanism to grow the right components of ∇u as ω itself is growing.** It is not difficult to construct situations where ω grows for all time but for which the "right component" of ∇u remains bounded for all time.
- (Stability) **Ideally, the mechanism should be robust enough to not be inhibited by small perturbations.** This is because in a non-local system one can never have "full" control on the global dynamics of a solution.

We can draw from the first and second points that a *fundamental* step toward understanding the finite-time singularity problem is to understand the mapping:

$$\mathcal{K} : \omega \to \nabla u,$$

of course taking into account that $\mathrm{div}(u) = 0$. This mapping is called the **Biot-Savart Law** and this is what we will study first.

2 The Biot-Savart Law (or, Elliptic regularity)

As discussed above, our first task is to study the Biot-Savart Law. We start with the two-dimensional case. Again, we are trying to find the map

$$\mathcal{K} : \omega \to \nabla u,$$

where

$$\omega = \nabla^{\perp} \cdot u \qquad \mathrm{div}(u) = 0.$$

Because $\mathrm{div}(u) = 0$, we can write:

$$u = \nabla^{\perp}\psi,$$

where ψ is called the stream function. Using the fact that $\omega = \nabla^{\perp} \cdot u$, we see that:

$$\Delta\psi = \omega.$$

Thus

$$u = \nabla^{\perp}\Delta^{-1}\omega = \nabla_x^{\perp} \int_{\mathbb{R}^2} K(x-y)\omega(y),$$

where K is the Newtonian potential. Thus,

$$u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2}\omega(y)dy.$$

Viewing the Biot-Savart law as an integral operator has many uses, but it is sometimes useful to simply view it as the solution to the system:

$$\Delta\psi = \omega \qquad u = \nabla^{\perp}\psi.$$

This point of view is particularly useful for L^2 based estimates such as the fact that

$$|\nabla u|_{H^s} = |\omega|_{H^s},$$

where we define

$$|f|_{H^s}^2 = \sum_{|\alpha| \leq s} \int |D^{\alpha} f|_{L^2}^2.$$

Note that this result follows from the $s = 0$ case by linearity, while the $s = 0$ case follows by integration by parts (if you have not done this calculation before, you should do it as an exercise). A natural question one could ask is whether we have pointless estimates. Namely, is it true that:

$$|\nabla u|_{L^\infty} \leq C|\omega|_{L^\infty}?$$

The answer is no. Indeed, let us suppose that $\omega \in C_c^\infty(\mathbb{R}^2)$ and $\omega(0) = 0$. Then, it is easy to see that

$$\nabla u(0) = \int_{\mathbb{R}^2} K(y)\omega(y)dy,$$

where

$$K(y) = -\frac{1}{2\pi|y|^4} \begin{bmatrix} y_1 y_2 & y_1^2 \\ -y_2^2 & -y_1 y_2. \end{bmatrix}$$

In particular, K is -2 homogeneous so it is easy to see that $\nabla u(0)$ can be very large even if $|\omega|_{L^\infty} \leq 1$.

Remark 2.1. The classical theory of singular integral operators, in fact, tells us that the above behavior also happens in L^p spaces:

Theorem 3 (Calderón-Zygmund)**.** *For $1 < p < \infty$, there is a constant C_p so that*

$$|\nabla u|_{L^p} \leq C_p|\omega|_{L^p}.$$

We will neither prove this theorem nor use it, but it is important to mention. What is important to note is that the result *does not* hold on L^∞ and, in fact, as $p \to \infty$ the constant $C_p \approx p$.

Punch-line: We will try to observe that as $p \to \infty$ it is possible to decompose ∇u into a singular part, which is easy to characterize, and a regular part which will be uniformly bounded as $p \to \infty$. That is, we will try to decompose:

$$\mathcal{K} = \mathcal{S} + \mathcal{R},$$

where $\mathcal{R}$ is a bounded operator independent of p and $\mathcal{S}$ is "easy" to understand. We will in fact only do this for certain types of functions ω.

2.1 Examples illustrating the behavior on L^∞

2.1.1 Radial Case

Observe that if

$$\psi(x) = |x|^\beta,$$

then

$$\Delta\psi = (\partial_{rr} + \frac{1}{r}\partial_r)(r^\beta) = \beta^2|x|^{\beta-2}.$$

Thus, up to a harmonic part, the solution to

$$\Delta\psi = |x|^{-\gamma}$$

is

$$\psi = \frac{1}{(2-\gamma)^2}|x|^{2-\gamma},$$

and we are interested in the case $\gamma \to 0$. Thus we see that:

$$|D^2\psi| \le C|x|^{-\gamma},$$

for C independent of γ as $\gamma \to 0$. It is in fact possible to show that $\mathcal{K}$ restricted to radial functions is bounded on L^p *independent* of p and, in particular, on L^∞.

2.1.2 General homogeneous vorticity

Since we did not discover the singular part for radial data, let us look at more general negative homogeneous data:

$$\psi(r,\theta) = r^\beta m(\theta),$$

where we leave m free for now. Then we see that

$$\Delta\psi(r,\theta) = (\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta})\psi = \Big(\beta^2 m(\theta) + m''(\theta)\Big)r^{\beta-2}.$$

At the same time,

$$|D^2\psi| \approx (|m| + |m'| + |m''|)r^{\beta-2}.$$

Thus, singular behavior occurs only when we have

$$|\beta^2 m + m''| << |m| + |m'| + |m''|.$$

This only occurs when: β is, or is very close to, some $n \in \mathbb{N}$ *and* when $\beta^2 m(\theta) + m''$ is very small. For example,

$$\psi(r,\theta) = r^\beta \sin(2\theta),$$

and β close to 2 gives:

$$\Delta\psi(r,\theta) = (4-\beta^2)\sin(2\theta)r^{2-\beta}.$$

Thus, as $\beta \to 2$, we see that $|D^2\psi| = O(1)$ but $\Delta\psi = O(2-\beta)$.

Remark 2.2. This result has been put in a quite general context in various works with In-Jee Jeong. The most general, perhaps, is in the work "On Singular Vortex Patches, I." For these lectures, we are not interested in establishing things in full generality.

2.2 General L^2 approach

We now move to understand the mapping $\mathcal{K}$ for a particular *type* of vorticity:

$$\omega(r,\theta) = \Omega(R,\theta),$$

where

$$R = r^\alpha,$$

and $\alpha > 0$ is **small.** Note that even if Ω is smooth, ω will generally only be C^α in space (though it may be C^∞ away from the origin). Now,

$$\mathcal{K} = D^2\Delta^{-1}.$$

So, we want to study the solution ψ of

$$\Delta\psi = \omega,$$

$$\partial_{rr}\psi + \frac{1}{r}\partial_r\psi + \frac{1}{r^2}\partial_{\theta\theta}\psi = \omega.$$

Now let us make the change:

$$\psi(r,\theta) = r^2\Psi(R^\alpha,\theta), \qquad \omega(r,\theta) = \Omega(R,\theta).$$

Thus,

$$\alpha^2R^2\partial_{RR}\Psi + (4\alpha+\alpha^2)R\partial_R\Psi + 4\Psi + \partial_{\theta\theta}\Psi = \Omega \tag{2.1}$$

2.2.1 The Regular Part

We begin with a discussion of the regular part of $\mathcal{K}$ on Ω.

Theorem 4. *Assume that*

$$\int_0^{2\pi} \Omega(R,\theta)\exp(in\theta)d\theta = 0,$$

for all R and for $n = 0, 1, 2$. Then, given $\Omega \in H^s$, there is a unique solution to (2.1) *satisfying*

$$\alpha^2|R^2\partial_{RR}\Psi|_{H^s} + \alpha|R\partial_R\Psi|_{H^s} + |\partial_{\theta\theta}\Psi|_{H^s} \le C_s|\Omega|_{H^s},$$

with the constant C_s **independent** *of α.*

Proof. We will only prove the L^2 estimate. The case $s \ge 1$ is easier. First, observe that Ψ also satisfies the orthogonality conditions. Indeed, if we define

$$\Psi_n(R) = (\Psi(R,\theta), \exp(in\theta))_{L^2_\theta}.$$

We see that

$$\alpha^2R^2\partial_{RR}\Psi_n + (4\alpha+\alpha^2)R\partial_R\Psi_n + (4-n^2)\Psi_n = 0.$$

Consequently,

$$\Psi_n(R) = c_{1,n}R^{\lambda_1} + c_{2,n}R^{\lambda_2},$$

where λ_1, λ_2 are the roots of

$$\alpha^2\lambda(\lambda-1) + (4\alpha+\alpha^2)\lambda + (4-n^2) = 0,$$

which is

$$\alpha^2\lambda^2 + 4\alpha\lambda + (4 - n^2) = 0.$$

Thus,

$$\lambda = \frac{-4\alpha \pm \sqrt{16\alpha^2 - 4\alpha^2(4 - n^2)}}{2\alpha^2}.$$

It is then easy to see that since we want $\Psi \in L^2$, we need

$$\Psi_n \equiv 0,$$

for $n = 0, 1, 2$. First, take

$$\alpha^2 R^2 \partial_{RR}\Psi + (4\alpha + \alpha^2)R\partial_R\Psi + 4\Psi + \partial_{\theta\theta}\Psi = \Omega,$$

multiply by $\partial_{\theta\theta}\Psi$, and integrate. Then we see:

$$|\partial_{\theta\theta}\Psi|_{L^2}^2 - 4|\partial_\theta\Psi|_{L^2}^2 + \int \alpha^2 R^2 \partial_{RR}\Psi\partial_{\theta\theta}\Psi + (4\alpha + \alpha^2)\int R\partial_R\Psi\partial_{\theta\theta}\Psi \leq |\Omega|_{L^2}|\partial_{\theta\theta}\Psi|_{L^2}.$$

The crucial point is that because of the orthogonality condition, we have that

$$|\partial_\theta\Psi|_{L^2}^2 \leq \frac{1}{9}|\partial_{\theta\theta}\Psi|_{L^2}^2.$$

Moreover,

$$(4\alpha + \alpha^2)\int R\partial_R\Psi\partial_{\theta\theta}\Psi = -(4\alpha + \alpha^2)\int R\partial_R\partial_\theta\Psi\partial_\theta\Psi = \frac{4\alpha + \alpha^2}{2}\int(\partial_\theta\Psi)^2.$$

Moreover,

$$\int \alpha^2 R^2 \partial_{RR}\Psi\partial_{\theta\theta}\Psi = -\alpha^2\int R^2\partial_{RR\theta}\Psi\partial_\theta\Psi = \alpha^2\int R^2|\partial_{R\theta}\Psi|^2 + 2\alpha^2\int R\partial_{R\theta}\Psi\partial_\theta\Psi = \alpha^2\int R^2|\partial_{R\theta}\Psi|^2 - \alpha^2\int(\partial_\theta\Psi)^2.$$

Thus we get:

$$\frac{1}{2}|\partial_{\theta\theta}\Psi|_{L^2}^2 + \alpha^2\int R^2|\partial_{R\theta}\Psi|^2 + \frac{4\alpha - \alpha^2}{2}\int(\partial_\theta\Psi)^2 \leq |\Omega|_{L^2}|\partial_{\theta\theta}\Psi|_{L^2}$$

The prove the rest, one should just multiply by $R^2\partial_{RR}\Psi$ and integrate. This is left as an exercise.

□

2.2.2 The Singular Part

Obviously, the singular part must come from functions which are *not* orthogonal to $\exp(in\theta)$ for some $n \in \{0, 1, 2\}$.

$$\alpha^2 R^2 \partial_{RR}\Psi + (4\alpha + \alpha^2)R\partial_R\Psi + 4\Psi + \partial_{\theta\theta}\Psi = \Omega$$

Theorem 5. *Assume that*

$$\Omega(R, \theta) = F(R)\exp(2i\theta)$$

and that $F \in L^2$. Then, the unique solution to (2.1) *with $\Psi \in L^2$ is*

$$\Psi(R, \theta) = \frac{1}{4\alpha}L(F)(R)\exp(2i\theta) + \mathcal{R}(\mathcal{F}),$$

where

$$L(F)(R) = \int_R^\infty \frac{F(s)}{s}ds$$

and

$$\|\mathcal{R}\|_{H^s \to H^s} \leq C_s,$$

independent of α.

Proof. As in the proof above, we can write:

$$\Psi(R,\theta) = G(R)\exp(2i\theta)$$

so that G satisfies:

$$\alpha^2 R^2 G''(R) + (4\alpha + \alpha^2) R G'(R) = F(R).$$

Now we just have to solve this ODE. Observe that

$$G'' + \frac{4+\alpha}{\alpha R} G'(R) = \frac{F}{\alpha^2 R^2}$$

Thus,

$$\left(G' R^{\frac{4+\alpha}{\alpha}}\right)' = R^{\frac{4+\alpha}{\alpha}} \frac{F}{\alpha^2 R^2}$$

Thus,

$$G'(R) = R^{-\frac{4+\alpha}{\alpha}} \int_0^R s^{\frac{4+\alpha}{\alpha}} \frac{F(s)}{\alpha^2 s^2} ds.$$

Thus,

$$\begin{aligned} G(R) &= \int_R^\infty \tau^{-\frac{4+\alpha}{\alpha}} \int_0^\tau s^{\frac{4+\alpha}{\alpha}} \frac{F(s)}{\alpha^2 s^2} ds. \\ &= -\int_R^\infty \frac{\alpha}{4} \frac{d}{d\tau} \tau^{-\frac{4}{\alpha}} \int_0^\tau s^{\frac{4+\alpha}{\alpha}} \frac{F(s)}{\alpha^2 s^2} ds \\ &= \frac{1}{4\alpha} \int_R^\infty \frac{F(s)}{s} ds + \frac{1}{4\alpha} R^{-\frac{4}{\alpha}} \int_0^R s^{\frac{4}{\alpha}} \frac{F(s)}{s} ds. \\ &= \frac{1}{4\alpha} L(F) + \mathcal{R}(F). \end{aligned}$$

□

Lemma 2.3. *There is a constant C independent of $0 \leq \alpha \leq 1$ so that*

$$|\mathcal{R}(F)|_{L^2} \leq C|F|_{L^2}.$$

We can take $C = \frac{1}{56}$.

Proof. The proof is similar to the proof of the Hardy Inequality.

$$\begin{aligned} \int_0^\infty \mathcal{R}(F)F &= \frac{1}{4\alpha} \int_0^\infty F(R) R^{-\frac{4}{\alpha}} \int_0^R s^{\frac{4}{\alpha}} \frac{F(s)}{s} ds = \frac{1}{4\alpha} \int_0^\infty R^{-\frac{8}{\alpha}+1} R^{4/\alpha} \frac{F(R)}{R} \int_0^R s^{4/\alpha} \frac{F(s)}{s} ds \\ &= \frac{1}{8\alpha} \int_0^\infty R^{-\frac{8}{\alpha}+1} \frac{d}{dR} \Big(\int_0^R s^{4/\alpha} \frac{F(s)}{s} ds \Big)^2 \\ &= \frac{1}{8\alpha} \Big(\frac{8}{\alpha} - 1 \Big) \int_0^\infty R^{-8/\alpha} \Big(\int_0^R s^{4/\alpha} \frac{F(s)}{s} ds \Big)^2 = \frac{64\alpha^2(8-\alpha)}{8\alpha^2} |\mathcal{R}(F)|_{L^2}^2. \end{aligned}$$

Thus,

$$|\mathcal{R}(F)|_{L^2} \leq \frac{1}{8(8-\alpha)} |F|_{L^2}.$$

□

3 Self-Similar Singularity

In the preceding section, we were able to show that under a certain change of variables (i.e. by looking at the correct type of functions), we can re-write the Euler equation as follows:

$$\partial_t \omega + u \cdot \nabla \omega = \nabla u \omega.$$

$$u = \frac{1}{\alpha} u_{\mathcal{S}} + u_{\mathcal{R}},$$

for α small. In particular, by scaling time by α, we see that

$$\partial_t \omega + u_{\mathcal{S}} \cdot \nabla \omega = \nabla u_{\mathcal{S}} \omega + \alpha N(u_{\mathcal{R}}, \omega),$$

where $u_{\mathcal{S}}$ has a very simple angular dependence and can be thought of as a function of one variable (the radial variable). Thus, for the problem when $\alpha = 0$, there is some hope to actually establish a finite-time singularity. The purpose of this lecture is to discuss, on toy examples, what kind of techniques one could use to pass from a blow-up when $\alpha = 0$ to a blow-up for $\alpha > 0$. I will discuss two techniques to approach this problem:

- Fixed point method,
- Compactness method.

3.1 Fixed point method

For an example of the fixed point approach, see previous work with In-Jee Jeong entitled "On the Effects of Advection and Vortex Stretching." Consider the following problem for $(x, t) \in [0, \infty) \times [0, \infty)$:

$$\partial_t f = f^2 + \epsilon N(f),$$

where N could be a functional satisfying the following propeties:

$$N(af) = a^2 N(f) \qquad N(f(a\,\cdot)) = N(f)(a\,\cdot),$$

for $a \in \mathbb{R}$. Let us also assume that N is a bounded operator, for example, on H^k for some $k \in \mathbb{N}$.

3.1.1 The case $\epsilon = 0$

When $\epsilon = 0$, we have:

$$\partial_t f = f^2.$$

This has many exact self-similar profiles:

$$f(x,t) = \frac{1}{1-t} F\Big(\frac{x}{1-t}\Big).$$

These are solutions to

$$F + z\partial_z F = F^2,$$

from which it is easy to see that $F \equiv 1$ is a solution (though this is quite unstable). The stable solution is:

$$F_0(z) = \frac{1}{1+z}.$$

3.1.2 The case of ϵ small

So, let us try to find a solution like this when $\epsilon > 0$. For this problem

$$\partial_t f = f^2 + \epsilon N(f),$$

let us search for a solution of the form:

$$f(x,t) = \frac{1}{1-t} F_\epsilon\Big(\frac{x}{(1-t)^{1+\delta(\epsilon)}}\Big),$$

where $\delta(\epsilon)$ is to be determined. Then,

$$F_\epsilon + (1+\delta_\epsilon) z\partial_z F_\epsilon = F_\epsilon^2 + \epsilon N(F_\epsilon).$$

Thus, letting $F_\epsilon = g + F_0$ we see:

$$g + z\partial_z g - \frac{2}{1+z} g = -\delta_\epsilon z \partial_z F_0 + g^2 + \epsilon N(F_0 + g),$$

which we write as:

$$\mathcal{L}(g) = -\delta_\epsilon z \partial_z F_0 + g^2 + \epsilon N(F_0 + g),$$

Lemma 3.1. *Let $f \in H^k$, $k \geq 2$. $\mathcal{L}(g) = f$ is solvable in C^1 if and only if $f'(0) + 2f(0) = 0$. Moreover, in this case we can write*

$$g(z) = -f(0) + \frac{z}{(z+1)^2} \int_0^z \frac{(t+1)^2}{t^2} \big(f(t) + f(0)\frac{t-1}{t+1}\big) dt.$$

and

$$|g|_{H^k} \leq C_k |f|_{H^k}.$$

Remark 3.2. Assuming the truth of this Lemma, the idea is just to use δ_ϵ to make sure that the right hand side satisfies the solvability condition.

Proof. Evaluating $\mathcal{L}(g) = f$ and its derivative at $z = 0$ gives

$$-g(0) = f(0),$$

$$2g(0) = f'(0).$$

This is what gives us the condition

$$f'(0) + 2f(0) = 0.$$

When f satisfies this, let us write:

$$\frac{z-1}{z+1} g + z\partial_z g = f,$$

so that

$$\frac{z-1}{z+1}(g - g(0)) + z\partial_z (g - g(0)) = f + f(0)\frac{z-1}{z+1}.$$

Let

$$G = g - g(0) \qquad F = f + f(0)\frac{z-1}{z+1}.$$

Observe that

$$F(0) = F'(0) = 0$$

using the condition.

$$\frac{z-1}{z+1}G + z\partial_z G = F.$$

Thus,

$$\frac{(z-1)}{(z+1)z}G + \partial_z G = \frac{F}{z}.$$

Note that

$$\int \frac{z-1}{(z+1)z} = \int \frac{2z-z-1}{(z+1)z} = 2\log(z+1) - \log(z).$$

Thus,

$$\partial_z\Big(\frac{(z+1)^2}{z}G\Big) = \frac{(z+1)^2}{z^2}F.$$

Thus,

$$G(z) = \frac{z}{(z+1)^2}\int_0^z \frac{(t+1)^2}{t^2}F(t)dt.$$

where we chose that $G'(0) = 0$ (since we have one compatibility condition, we have one free condition on G and this is how we chose it). Thus,

$$g(z) = -f(0) + \frac{z}{(z+1)^2}\int_0^z \frac{(t+1)^2}{t^2}F(t)dt.$$

The proof of the H^k bound follows from small generalizations of the Hardy inequality and we leave it as an exercise. There may be some concern about the $f(0)$ terms for L^2 bounds so we will explain this part. Let us observe that when $z \geq 1$,

$$\begin{aligned} -f(0) + \frac{z}{(1+z)^2}\int_1^z \frac{(t+1)^2}{t^2}F(t)dt &= -f(0) + \frac{z}{(1+z)^2}\int_1^z \frac{(t+1)^2}{t^2}f(0)\frac{t-1}{t+1}dt \\ &= f(0)\Big(-1 + \frac{z}{(1+z)^2}\int_1^z \frac{t^2-1}{t^2}dt\Big) = f(0)\mathrm{O}(\frac{1}{z}). \end{aligned}$$

□

Now, back to finding g:

$$\mathcal{L}(g) = -\delta_\epsilon z\partial_z F_0 + g^2 + \epsilon N(F_0 + g),$$

Observe that the compatibility condition is:

$$2g(0)^2 + 2\epsilon N(F_0+g)(0) - \delta_\epsilon F_0'(0) + 2g(0)g'(0) + \epsilon N(f_0+g)'(0) = 0.$$

Since $F_0'(0) = 1$ we thus set:

$$\delta_\epsilon = 2g(0)^2 + 2\epsilon N(F_0+g)(0) + 2g(0)g'(0) + \epsilon N(f_0+g)'(0)$$

Then we can write:

$$g = \mathcal{L}^{-1}\Big(-\delta_\epsilon z\partial_z F_0 + g^2 + \epsilon N(F_0+g)\Big) := \mathcal{K}(g).$$

Theorem 6. *Since N is bounded on H^k for some $k \geq 2$, there is a $\delta > 0$ (which is just a constant multiple of ϵ) so that*

$$\mathcal{K} : \overline{B_\delta(0)} \to \overline{B_\delta(0)}$$

is a contraction (where $B_\delta(0)$ is the ball of radius δ around 0 in H^k).

Remark 3.3. We don't actually need that N is bounded on H^k. All we need is that $\mathcal{L}^{-1}(N)$ is bounded on H^k, which is possible in some cases even when N contains derivatives, though I am only aware that this can be done for one-dimensional problems. The fixed point approach can be seen as like the implicit function theorem. For higher dimensional problems with derivatives, we need the compactness method which we turn to next.

3.2 Compactness method

The compactness approach follows essentially the same line of reasoning as the fixed point approach but instead of using invertibility of the linearized operator, we use its ellipticity on certain spaces. The point of this is to deal with possible unboundedness of N in

$$\partial_t f = f^2 + \epsilon N(f)$$

3.2.1 General philosophy

Suppose we are trying to solve an equation of the form:

$$L(f) = g + \epsilon N(f).$$

When $\mathcal{L}^{-1}N$ is a bounded operator, we can do as above and solve it via a clear iteration scheme. In the case that L is a positive operator and N satisfies:

$$|(N(f), f)| \leq C|f|^2,$$

while

$$(L(f), f) \geq c|f|^2,$$

we are still OK. Indeed, in this case we can add a fake time variable τ :

$$\partial_\tau f + L(f) = g + \epsilon N(f).$$

In the above, I am thinking that $(,)$ is like the H^k inner-product or similar.

Since g is independent of τ, it is usually then possible to show that $\partial_\tau f$ decays exponentially in, say H^{k-1} while f is bounded in H^k. The existence of a solution then follows.

3.2.2 Details

Let us go into the details.

$$\partial_t f = f^2 + \epsilon N(f),$$

Let's now search for a solution of the form

$$f(x,t) = \frac{1}{1-(1+\mu(\epsilon))t} F_\epsilon\Big(\frac{x}{(1-(1+\mu(\epsilon))t)^{1+\delta(\epsilon)}}\Big).$$

Essentially the point we will see is that while for invertibility of $\mathcal{L}$ we needed only one free parameter, we will need two to establish ellipticity. Then we get:

$$(1+\mu(\epsilon))F_\epsilon + z(1+\mu(\epsilon))(1+\delta(\epsilon))\partial_z F_\epsilon = F_\epsilon^2 + \epsilon N(F_\epsilon).$$

Now write:

$$F_\epsilon = F_0 + g.$$

Then,

$$g + z\partial_z g - \frac{2}{1+z} g = -\mu F_0 - (\mu + \lambda + \mu\lambda) z\partial_z F_0 + g^2 - \mu g - (\mu + \lambda + \mu\lambda) z\partial_z g + \epsilon N(F_0 + g).$$

Thus,

$$\mathcal{L}(g) = -\mu F_0 - (\mu + \lambda + \mu\lambda) z\partial_z F_0 + g^2 - \mu g - (\mu + \lambda + \mu\lambda) z\partial_z g + \epsilon N(F_0 + g). \tag{3.1}$$

Our goal is now to find a space X so that

$$(\mathcal{L}(g), g)_X \geq c|g|_X^2 \tag{3.2}$$

while

$$(RHS, g)_X \leq C\epsilon(|g|_X + |g|_X^2) + C|g|_X^3. \tag{3.3}$$

Recall that

$$\mathcal{L}(g) = g + z\partial_z g - \frac{2}{1+z}g.$$

Consider a g supported in $[0,1]$ which satisfies $g(0) > 0$ and then decreases monotonically to 0. Obviously, for such g, $(\mathcal{L}(g), g)_{L^2} < 0$, while for g supported at large z we see that $(\mathcal{L}(g), g) > 0$. This means that it will not be able to take the space X to be a normal Sobolev space.

If we wish to show that $\mathcal{L}$ is positive, we should try to penalize mass at 0. It turns out that if one just defines the weight

$$w(z) = \frac{(1+z)^4}{z^4},$$

then

$$(\mathcal{L}(f), f)_{L^2_w} = \frac{1}{2}|f|^2_{L^2_w}.$$

Note that it isn't really necessary to have have an exact weight which will give us equality as above, but the point is to take a strong weight near $z = 0$. There are now a number of ways to get around this issue. Choosing the weight for a given problem can be quite involved.

3.2.3 Choice of μ and λ

Since we are talking about taking L^2_w with weight $w = \frac{(1+z)^4}{z^4}$, it means that we need to ensure that g and the right hand side vanish quadratically at $z = 0$. Observe that, as in the fixed point approach, we can use μ and λ to satisfy these two conditions since we have on the RHS or (3.1) the term

$$-\mu F_0 - (\mu + \lambda + \lambda\mu)z\partial_z F$$

, which when evaluated at $z = 0$ gives μ and when differentiated and evaluated at 0 gives $2\mu + \lambda + \mu\lambda$. This means that we can choose μ to ensure $RHS(0) = 0$ and then choose λ to ensure $RHS'(0) = 0$.

3.2.4 Definition of inner product

Lemma 3.4. *It is possible to find constants $c_1, c_2 > 0$ so that the inner product*

$$(f, g)_X = (f, g)_{L^2_w} + c_1(z\partial_z f, z\partial_z g)_{L^2_w} + c_2((z\partial_z)^2 f, (z\partial_z)^2 g)_{L^2_w}$$

is such that (3.2) *hold.*

Remark 3.5. Additionally, (3.3) holds with the above choices of μ and λ so long as N satisfies reasonable assumptions (basically that we can do standard energy estimates).

Corollary 3.6. *(A-priori estimate) There exists a $C_* > 0$ and an $\epsilon > 0$ sufficiently small so that if a solution to* (3.1) *$g \in X$ exists with $|g|_X \leq C_*\epsilon$, then actually $|g|_X \leq C_*\frac{\epsilon}{2}$.*

4 Application I: Axi-symmetric solutions without swirl

Up to now, we have discussed two concepts:

- How to simplify the non-local effects in the Euler equation by making the change $r \to r^\alpha$, where r is the radial variable.
- How to pass from a blow-up for one equation to the blow-up for a nearby equation assuming some linear stability.

Now we would like to apply these ideas to the 3d Euler equation along with one extra idea. The simplest case in which one can apply these ideas is the case of axi-symmetric solutions without swirl. To keep the discussion light, I will not discuss in detail the derivation of this equation from the original 3d Euler system, but it is a very good exercise.

First, we recall that whenever a matrix $\mathcal{O}$ satisfies

$$\mathcal{O}\mathcal{O}^t = Id,$$

then $u(x,t)$ solves Euler implies that

$$\mathcal{O}^t u(\mathcal{O}x, t)$$

is also a solution. Now we take

$$\mathcal{O}_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A solution is axi-symmetric if

$$u(\mathcal{O}_\phi x, t) = \mathcal{O}_\phi u(x,t),$$

for all ϕ, x, t. Reflecting upon this for a moment, u must take the following form:

$$u = u_r(r,z)\begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \\ 0 \end{pmatrix} + u_\phi(r,z)\begin{pmatrix} -\frac{x_2}{r} \\ \frac{x_1}{r} \\ 0 \end{pmatrix} + u_z(r,z)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where

$$r = \sqrt{x_1^2 + x_2^2}.$$

Furthermore the pressure must be of the form $p = P(r,z)$. Because of this last fact, it turns out that u_ϕ is simply transported by (u_r, u_z). This means that we may assume that

$$u_\phi \equiv 0.$$

This is the no-swirl condition. In this case, if we define

$$\omega = \partial_r u_z - \partial_z u_r,$$

$$U = (u_r, u_z)$$

we get that

$$\partial_t \omega + U \cdot \nabla_{r,z}\omega = \frac{u_r}{r}\omega,$$

$$\mathrm{div}(rU) = 0,\ \omega = \partial_r u_z - \partial_z u_r.$$

4.0.1 Boundary conditions and the issue of smoothness

The physical domain of the solution is:

$$(r,z) \in [0,\infty) \times \mathbb{R}.$$

When $r = 0$, the system does not actually require any boundary condition on ω. However, in order than the original u be smooth, ω_0 must vanish at least linearly on $r = 0$. Of course, if we only care that $u \in C^{1,\alpha}$ all it means is that ω must vanish like r^α as $r \to 0$. The system does impose a boundary condition on u_r. We need u_r to vanish at $r = 0$ linearly (just for the solution to be C^1).

Question: In this setting, how do we ensure growth of ω?

In this particular setting, it is easy to see that in order for ω to grow, we need to be working in a scenario where $\frac{u_r}{r} > 0$ (at least near the maximum of ω). It is easy to see that if we put an odd symmetry in z and also make ω positive in $[0,\infty) \times [0,\infty)$, u_r will generally be positive, at least on $z = 0$.

Two problems

- Non-locality, stability, etc. ✓
- Stabilizing effect of transport term.

In fact, the stabilizing effect of the transport term is so strong that it prohibits a blow-up for sufficiently smooth data. Indeed, we can write:

$$\partial_t \frac{\omega}{r} + U \cdot \nabla \frac{\omega}{r} = 0.$$

Hence, we have uniform bounds on $\frac{\omega}{r}$ when the initial data is smooth. The idea is that when ω vanishes along $r = 0$, if we have $u_r > 0$ (as we should for a blow-up), the transport term will push the small values of ω to invade the whole domain. This is the stabilizing effect of the transport term. Even for $C^{1,\alpha}$ solutions, we will see that there is a piece of the transport term that will try to fight the blow-up.

4.1 Passage to the fundamental model

Just as we mentioned for the 2d Laplacian, now if we introduce

$$R = \rho^\alpha, \theta = \arctan(\frac{z}{r})$$

where $\rho = \sqrt{r^2 + z^2}$ we observe that the system becomes:

$$\partial_t \Omega - \frac{3}{2} L_{12}(\Omega) \sin(2\theta) \partial_\theta \Omega = L_{12}(\Omega)\Omega + \alpha N(\Omega),$$

$$L_{12}(\Omega)(R) = \int_R^\infty \int_0^{\pi/2} \frac{\Omega(s,\theta)}{s} K(\theta) d\theta ds,$$

where $K(\theta) = 3\cos^2(\theta)\sin(\theta)$.

Let us now set $\alpha = 0$**.** Then we see that

$$\partial_t \Omega - \frac{3}{2} L_{12}(\Omega) \sin(2\theta) \partial_\theta \Omega = L_{12}(\Omega)\Omega. \tag{4.1}$$

$$L_{12}(\Omega)(R) = \int_R^\infty \int_0^{\pi/2} \frac{\Omega(s,\theta)}{s} K(\theta) d\theta ds. \tag{4.2}$$

This is the system

$$\partial_t \omega + u_{\mathcal{S}} \cdot \nabla \omega = \nabla u_{\mathcal{S}} \omega,$$

which we mentioned before, restricted to axi-symmetric solutions without swirl.

It is not difficult to check that (4.1)-(4.2) is, in fact, globally regular for data which is smooth in θ and R and vanishing on $\theta = 0$.

Lemma 4.1. *Assume that $\Omega_0 \in C_c^{1/3}([0, \frac{\pi}{2}] \times [0,\infty))$ and $\Omega_0(R, 0) = 0$, then the unique local solution of (4.1)-(4.2) is global. If, however, $\Omega_0(\theta, R) \geq (\pi/2 - \theta)^\alpha$ for θ near $\pi/2$ and for any $\alpha < \frac{1}{3}$, the solution can blow-up in finite time.*

4.2 Dropping the Angular transport term for solutions that are almost θ independent

When Ω is independent of θ, (4.1)-(4.2) becomes:

$$\partial_t \Omega(R,t) = \Omega(R,t) \int_R^\infty \frac{\Omega(s,t)}{s} ds.$$

This is like a non-local Ricatti equation. It turns out that very similar arguments to what we described in Section 3 above (the compactness approach), yield that the self-similar blow-up profile:

$$\Omega(R,t) = \frac{1}{1-t} F(\frac{R}{1-t}), \qquad F(z) = \frac{z}{(1+z)^2}, \qquad L_{12}(F) = \frac{1}{(1+z)}$$

is stable.

4.3 Analysis of the linearized operator $\mathcal{L}$ and design of angular weights

When interpreted as a solution of (4.1)-(4.2), the linearized operator becomes:

$$\mathcal{L}g = g + z\partial_z g - \frac{2}{1+z} g - \frac{z}{(1+z)^2} L_{12}(g) + \frac{3}{2(1+z)} \sin(2\theta)\partial_\theta g.$$

The way to deal with $\mathcal{L}$, we have to be very careful in how we deal with the angular transport term. Our goal is to design a reasonable inner product space X for which we have

$$(\mathcal{L}g, g)_X \geq c|g|_X^2.$$

Let us adopt the notation:

$$D_\theta = \sin(2\theta)\partial_\theta, \qquad D_z = z\partial_z.$$

Now, observe that

$$D_\theta \mathcal{L}g = D_\theta g + z\partial_z D_\theta g - \frac{2}{1+z} D_\theta g + \frac{3}{2(1+z)} \sin(2\theta)\partial_\theta (D_\theta g).$$

We already have a weight

$$w_z(z) = \frac{(1+z)^4}{z^4}$$

which gives us positivity from the first three terms. For the last term, we can simply add the weight

$$w_\theta = \frac{1}{\sin(2\theta)^{1+\delta}},$$

and if δ is sufficiently small, we will see that

$$(D_\theta \mathcal{L}g, D_\theta g)_{L^2_w} \geq (\frac{1}{2} - \delta)|D_\theta g|^2_{L^2_w},$$

where

$$w = w_z w_\theta.$$

This allows us to get control of $D_\theta g$ in terms of itself. We then define an inner product of order 1 as follows:

$$(f,g)_{\mathcal{H}^1} = (D_\theta f, D_\theta g)_{L^2_w} + c_1 (f,g)_{L^2_w} + c_2 (D_z f, D_z g)_{L^2_w}$$

and c_1 can be taken small and then c_2 smaller so that we have:

$$(\mathcal{L}g, g) \geq c|g|^2_{\mathcal{H}^1}.$$

The choice of taking $\delta > 0$ is to ensure that $\mathcal{H}^k$ embeds in C^α for some $\alpha > 0$.

4.4 Final Remarks

A fundamental problem is now to try to upgrade the result to a blow-up for smoother solutions in other scenarios. Another path one could take is to try to consider the application of the methods described above to other problems as well as other types of questions.