Ceresa cycles of Fermat curves and Hodge theory of fundamental groups

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Relations on algebraic cycles

- Recall that one has (among others) the notions of rational, algebraic, and homological equivalence on the space of algebraic n-cycles on a smooth complex projective variety. Rationally (resp. algebraically) trivial cycles are generated by differences of cycles that can be deformed to one another along P^1 (resp. an algebraic curve). Homologically trivial cycles are those in the kernel of the class map; they are boundaries of topological chains.
- Notation: Let $\mathcal{Z}_n(X)_{\mathbb{Q}}$:= space of algebraic n-cycles on X with coefficients in $\mathbb{Q}$. Use superscripts *rat*, *alg*, and *hom* for the rationally, algebraically, and homologically trivial subspaces, respectively.
- $\mathcal{Z}_n^{rat}(X)_{\mathbb{Q}} \subset \mathcal{Z}_n^{alg}(X)_{\mathbb{Q}} \subset \mathcal{Z}_n^{hom}(X)_{\mathbb{Q}}$. In general, these inclusions are strict.

- Griffiths showed in 1969 that if X is a general quintic in P^4, then $\mathcal{Z}_1^{hom}(X)_{\mathbb{Q}}/\mathcal{Z}_1^{alg}(X)_{\mathbb{Q}}$ is not zero. Clemens (1983) showed that this is not even necessarily finitely generated (although it is countable).
- Let X be a curve, Jac the Jacobian of X, $e \in X$, $X_e \subset Jac$ the image of $x \mapsto x - e$ and X_e^- the image of X_e under the inversion map. Then $X_e - X_e^- \in \mathcal{Z}_1^{hom}(Jac)$. Ceresa showed in 1983 that for a generic curve of genus ≥ 3, $X_e - X_e^-$ is not algebraically trivial. (We call $X_e - X_e^-$ the Ceresa cycle of X with base point e.)

- The first explicit example of a homologically trivial but not algebraically trivial cycle was given by Bruno Harris (1983). By expressing the Abel-Jacobi image of the Ceresa cycle in terms of iterated integrals he was able to show that $X_e - X_e^-$ is algebraically nontrivial for $X = F(4)$, the Fermat curve of degree 4 (defined by $X^4 + Y^4 = Z^4$).
- Soon after Harris' proof, Bloch showed using the ℓ-adic Abel-Jacobi map that the Ceresa cycle of $F(4)$ is of infinite order modulo algebraic equivalence.
- More recently, Tadokoro and Otsubo have generalized Harris' method to get nontriviality results modulo algebraic equivalence for Ceresa cycles of some other curves (Otsubo: Fermat curves of degree ≤ 1000, Tadokoro: Klein quartic curve, cyclic quotients of Fermat curves). Here one finds a sufficient condition for nontriviality of the the Ceresa cycle modulo algebraic equivalence in terms of non-integrality of an integral (which can be verified numerically for specific examples).

The goal of this talk is to prove the following

Theorem (PE - K. Murty)

The Ceresa cycles of Fermat curves of prime degree $p > 7$ are of infinite order modulo rational equivalence.

The proof combines several results on the Hodge theory of the space of quadratic iterated integrals on a curve with some number theoretic results of Gross and Rohrlich.

Plan of the rest of the talk

1. Reminder on the Abel-Jacobi map
2. Reminder on Ext groups in the category of mixed Hodge structures
3. Hodge theory of π_1
4. Proof of the result
5. Final remarks (on obtaining results modulo algebraic equivalence)

Review of Abel-Jacobi maps

- For a (rational) Hodge structure H of odd weight $2n-1$, define the (Griffiths) intermediate Jacobian of H (mod torsion) to be
$$JH := \frac{H_{\mathbb{C}}}{F^n H_{\mathbb{C}} + H_{\mathbb{Q}}}.$$
(When H is integral, this is the compact complex torus $\frac{H_{\mathbb{C}}}{F^n H_{\mathbb{C}} + H_{\mathbb{Z}}}$ modulo its torsion sobgroup.)

- Let X be a smooth projective variety over $\mathbb{C}$. The Abel-Jacobi map of Griffiths (tensored with $\mathbb{Q}$) is the map

$$AJ : CH_n^{\text{hom}}(X)_{\mathbb{Q}} := \mathcal{Z}_n^{hom}(X)_{\mathbb{Q}} \,/\, \mathcal{Z}_n^{rat}(X)_{\mathbb{Q}} \longrightarrow JH_{2n+1}(X)$$

defined as follows: We have $JH_{2n+1}(X) \cong (F^{n+1}H^{2n+1}(X,\mathbb{C}))^{\vee}/H_{2n+1}(X,\mathbb{Q})$. Given $Z \in \mathcal{Z}_n^{\text{hom}}(X)_{\mathbb{Q}}$, pick a rational topological $(2n+1)$-cycle Γ such that $\partial\Gamma = Z$. Given a smooth closed $(2n+1)$-form ω in F^{n+1}, set $\int_\Gamma[\omega] := \int_\Gamma \omega$; this defines a functional $\int_\Gamma \in (F^{n+1}H^{2n+1}(X,\mathbb{C}))^{\vee}$. Set

$$AJ(Z) = [\int_\Gamma] \in (F^{n+1}H^{2n+1}(X,\mathbb{C}))^{\vee}/H_{2n+1}(X,\mathbb{Q})$$
$$\cong JH_{2n+1}(X).$$

Extentions of mixed Hodge structures

- Let A be a mixed Hodge structure (MHS) of weight $2n+1>0$. By a theorem of J. Carlson, there is a functorial isomorphism
$$Ext(A, \mathbb{Q}(-n)) \cong JA^{\vee}$$
(which sends $0 \longrightarrow \mathbb{Q}(-n) \xrightarrow{\iota} E \xrightarrow{\pi} A \longrightarrow 0$ to the class of $r \circ s$, where s is a section of $\pi : E_{\mathbb{C}} \longrightarrow A_{\mathbb{C}}$ compatible with the Hodge filtration and r is a retraction of $\iota : \mathbb{Q} \longrightarrow E_{\mathbb{Q}}$).
- Apply to $A = H^{2n+1}(X)$, we can identify
$$JH_{2n+1}(X) \cong Ext(H^{2n+1}(X), \mathbb{Q}(-n)).$$

Hodge theory of π_1

- Let X be a smooth (not necessarily projective) complex variety and $e \in X$. Let I be the augmentation ideal $\mathbb{Q}[\pi_1(X,e)] \longrightarrow \mathbb{Q}$. By the work of K. T. Chen, elements of $(I/I^{n+1})^\vee$ (and $(I/I^{n+1})^\vee \otimes \mathbb{C}$) can be described using closed iterated integrals of length $\leq n$.
- Using the description of $(I/I^{n+1})^\vee$ as iterated integrals, Hain defined a MHS on $(I/I^{n+1})^\vee$, which we denote by $L_n(X,e)$, functorial with respect to (X,e) and such that the inclusions $(I/I^{n+1})^\vee \subset (I/I^{n+2})^\vee$ are morphisms.
- $L_1(X,e) = H^1(X)$ is independent of e, but for $n > 1$ these depend on e.

- We are interested in the case $n = 2$. One has a morphism

$$L_2(X, e) \longrightarrow H^1(X) \otimes H^1(X) \tag{1}$$

which sends $[f] \in (I/I^3)^\vee$ to the following element of $H^1(X) \otimes H^1(X) \cong (H_1(X) \otimes H_1(X))^\vee$:

$$[\gamma_1] \otimes [\gamma_2] \mapsto f((\gamma_1 - 1)(\gamma_2 - 1)) \quad (\gamma_i \in \pi_1(X, e)).$$

- The kernel of Eq. (1) is $H^1(X)$ and its image is the kernel of the cup product $(H^1(X) \otimes H^1(X))'$.

The extensions E_e and E_e^∞

We now focus on the case of curves. From this point on, X is a smooth complex projective curve, $e, \infty \in X$ are distinct, and we write H^1 for $H^1(X) = H^1(X - \infty)$. By the previous slide we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & L_2(X, e) & \longrightarrow & (H^1 \otimes H^1)' & \longrightarrow & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \longrightarrow & H^1 & \longrightarrow & L_2(X - \{\infty\}, e) & \longrightarrow & H^1 \otimes H^1 & \longrightarrow & 0 \end{array}$$

with exact rows. We call the two extensions in the top and bottom rows E_e and E_e^∞, respectively. Thus

$$E_e \in Ext((H^1 \otimes H^1)', H^1) \cong Ext(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Q}(-1))$$
$$E_e^\infty \in Ext(H^1 \otimes H^1, H^1) \cong Ext(H^1 \otimes H^1 \otimes H^1, \mathbb{Q}(-1))$$

(identifications via Poincaré duality).

Proof of the theorem

- The proof is an application of the combination of following:
  - Works of B. Harris and Pulte on the relation between the Ceresa cycle and E_e (1980's)
  - Works of Kaenders (2001) and Darmon-Rotger-Sols (2012) on E_e^∞ of a punctured curve
  - A result of Gross-Rohrlich on points of infinite order on the Jacobian of Fermat curves (1978)
  - A well-known result of Rohrlich on points on the Jacobian of Fermat curves which are supported on the cusps (1977)
- We will prove that the Abel-Jacobi image of the Ceresa cycle of the Fermat curve $F(p)$ of prime degree $p > 7$ is nonzero. (Recall that our Abel-Jacobi maps are tensored with $\mathbb{Q}$.)

A theorem of Darmon-Rotger-Sols

- Let $Z \in CH_1(X \times X)$. Let $\Delta \in CH_1(X \times X)$ be the diagonal of X. Set $Z_{12} = Z \cdot \Delta$, considered as an element of $CH_0(X)$. Let $P_Z = Z_{12} - \deg(Z_{12})e \in CH_0^{hom}(X)$.
- The point P_Z is related to the extension E_e^∞ as follows: Denote by ξ_Z the $H^1 \otimes H^1$ Künneth component of the class of Z in $H^2(X \times X)$. Then pullback along the morphism $H^1(-1) \longrightarrow (H^1)^{\otimes 3}$ defined by $\omega \mapsto \omega \otimes \xi_Z$ gives a map

$$\begin{aligned} \xi_Z^{-1} : Ext((H^1)^{\otimes 3}, \mathbb{Q}(-1)) \longrightarrow & Ext(H^1, \mathbb{Q}(0)) \\ \cong & \ J(H^1)^\vee \\ \overset{\text{AJ}}{\cong} & \ CH_0^{\text{hom}}(X)_{\mathbb{Q}}. \end{aligned}$$

Theorem:

$$\xi_Z^{-1}(E_e^\infty) = (\int_\Delta \xi_Z)(\infty - e) - P_Z.$$

- We apply this to $X = F(p)$. Take ∞ and p to be cusps. We will show that E_e^∞ is nonzero (as an extension of rational mixed Hodge structures).
- Let Z be the graph of the automorphism of $F(p)$ sending

$$(x, y, z) \mapsto (-y, z, x).$$

- This automorphism has two fixed points, namely

$$Q = (\eta, \eta^{-1}, 1) \qquad \text{and} \qquad \overline{Q} = (\eta^{-1}, \eta, 1),$$

where η is a primitive sixth root of unity. Thus

$$P_Z = Q + \overline{Q} - 2e.$$

- By a theorem of Gross and Rohrlich for $p > 7$ this is a point of infinite order on the Jacobian of $F(p)$. By a result of Rohrlich, $\infty - e$ is torsion on the Jacobian. It follows that E_e^∞ is nonzero.

E_e^∞ and Ceresa cycles: Results of Harris-Pulte and Kaenders

- Let ξ_Δ be the $H^1 \otimes H^1$ component of the class of the diagonal Δ of X. We have a decomposition of Hodge structures

$$H^1 \otimes H^1 \otimes H^1 = H^1 \otimes (H^1 \otimes H^1)' \oplus H^1 \otimes \xi_\Delta.$$

- Theorem (Harris-Pulte): The restriction of E_e^∞ to $H^1 \otimes (H^1 \otimes H^1)'$ (i.e. E_e) is 1/2 times the image of $X_e - X_e^-$ under

$$\begin{aligned} CH_1^{hom}(Jac) \xrightarrow{AJ} & \quad JH_3(Jac) = J(\textstyle\bigwedge^3 H^1)^\vee \\ \hookrightarrow & \quad J\left(H^1 \otimes (H^1 \otimes H^1)' \right)^\vee \\ \cong & \quad Ext(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Q}(-1)). \end{aligned}$$

- Theorem (Kaenders): The restriction of E_e^∞ to $H^1 \otimes \xi_\Delta$ as a point in

$$Ext(H^1 \otimes \xi_\Delta, \mathbb{Q}(-1)) \cong Ext(H^1, \mathbb{Q}(0)) = J(H^1)^\vee \overset{\mathrm{AJ}}{\cong} CH_0^{hom}(X)$$

  is $-2g\infty + 2e + K$, where K is the canonical divisor.
- We apply these results to $X = F(p)$ with ∞, e cusps. Recall that then E_p^∞ is of nonzero. Being supported on the cusps, $-2g\infty + 2e + K$ is torsion on the Jacobian. It follows that $AJ(F(p)_e - F(p)_e^-)$ is nonzero.

Arbitrary base point

- So far we have shown the result if e is a cusp. To get it for arbitrary base point, we need two more results. Let us go back to the generality of an arbitrary curve. Write

$$\bigwedge^3 H^1 = (\bigwedge^3 H^1)_{prim} \oplus H^1 \wedge \overline{\xi}_\Delta,$$

  where the first summand is the primitive cohomology in $H^3(Jac)$ and $\overline{\xi}_\Delta$ is the image of ξ_Δ in $\bigwedge^2 H^1$, which is an integral Kähler class of Jac.
- Theorem (B. Harris): The restriction of $AJ(X_e - X_e^-)$ to the primitive cohomology is independent of the base point.

- Theorem (Pulte): The restriction of $AJ(X_e - X_e^-)$ to $H^1 \wedge \overline{\xi}_\Delta$ in
$$J(H^1 \wedge \overline{\xi}_\Delta)^\vee = JH^1 \cong CH_0^{hom}(X)$$
is $(2g-2)e - K$.
- Back to $X = F(p)$ with e still for the moment a cusp, $(2g-2)e - K$ is torsion on the Jacobian. So the restriction of $AJ(X_e - X_e^-)$ to the primitive part is nonzero for a cusp e, and hence by Harris, for arbitrary e.

Final remarks: working module algebraic equivalence

- Let Y be a smooth projective variety, $Z \in \mathcal{Z}_n^{hom}(Y)$ (coefficients in $\mathbb{Z}$). The (integral) Abel-Jacobi image is

$$AJ(Z) = [\int_{\partial^{-1}Z}] \in (F^{n+1}H^{2n+1}(Y,\mathbb{C}))^\vee / H_{2n+1}(Y,\mathbb{Z}) \ = \ JH_{2n+1}(Y)$$

  (here in defining $JH_{2n+1}(Y)$ we are modding out by $H_{2n+1}(Y,\mathbb{Z})$, rather than $H_{2n+1}(Y,\mathbb{Q})$).
- If Z is algebraically trivial, then $\int_{\partial^{-1}Z}$ vanishes on $F^{n+2}H^{2n+1}(Y,\mathbb{C})$. Thus one has a sufficient condition for nontriviality modulo algebraic equivalence.

- In the case of Ceresa cycles in Jacobians, Harris expressed the Abel-Jacobi image in terms of iterated integrals (recall $AJ(X_e - X_e^-) = 1/2E_e$). In the case of $F(4)$, he obtained a sufficient condition for algebraic nontriviality in terms of non-integrality of an iterated integral involving holomorphic forms, which can be verified numerically.
- More recently, Otsubo and Tadokoro have used versions of this argument to deal with some other cases. The last step of these arguments is always done numerically. Also, it is hard to prove results about having infinite order (as that cannot be done numerically).
- Here by using the result of Darmon-Rotger-Sols and Gross-Rohrlich we were able to avoid numerical verifications and we proved that the Ceresa cycle is of infinite order, although only modulo rational equivalence. Can one modify the argument to get a result modulo algebraic equivalence?