Extending the Theta Correspondence

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Plan of This Talk

Plan:

1. The Classical Theta Correspondence (a quick, informal review)
2. Beyond the Weil Representation
3. Non-Minimal Theta Correspondences
4. New Work, joint with David Ginzburg: Extending the Classical Theta Correspondence to Higher Degree Metaplectic Covers

The Classical Theta Correspondence: Early History

- Begins with the Shimura correspondence between half integral modular forms $f(z)$ and integral weight modular forms $f'(z')$; formulated adelically by Gelbart-Piatetski-Shapiro.
- Realized by theta series by Shintani and Niwa. Theta series allowed the construction of an integral kernel $\theta(z, z')$ such that

$$f'(z') = \int f(z)\,\overline{\theta}(z, z')\,dz.$$

- The modularity of the theta series is proved using Poisson summation.
- These theta series have Fourier coefficients "concentrated on squares," i.e. have many vanishing Fourier coefficients.
- The Shimura correspondence via theta functions was studied and analyzed, adelically, by Waldspurger (1979, 1980).

The Classical Theta Correspondence: Additional HIstory

More Generally: Let $G = SO(V)$, $G' = Sp(W')$, $W = V \otimes W'$ with symplectic form $< \ , \ >_V \otimes < \ , \ >_{W'}$. By taking a theta function on $Sp(W)$ (really its metaplectic double cover $Mp(W)$) and restricting it to $G \times G'$, one obtains a theta correspondence. The Shimura correspondence comes from $G = SO(3)$, $G' = Sp_2 = SL_2$. Since dim V is odd, one gets an automorphic form on the double cover, i.e. of half-integral weight.

Key group-theoretic feature:

Definition (Howe)

Suppose G and G' are reductive subgroups of a symplectic group $Sp(W)$ and each is the full centralizer of the other in $Sp(W)$. Then (G, G') is a *reductive dual pair*.

Then one can produce a (global) theta corrrespondence by restricting a theta function on (the double cover of) $Sp(W)$ to the embedded image of $G \times G'$ and using it as an integral kernel.

The Classical Theta Correspondence: Additional History II

Google Scholar: "Theta correspondence": About 271,000 entries.

- Theta towers (Rallis): Let $\mathbf{H}$ be the two dimensional hyperbolic quadratic space. Study the dual pairs $(SO(V + n\mathbf{H}), Sp(W))$ as n varies. If π is an automorphic representation of $Sp(W)$, show that there is a nonzero theta lift in this tower, and that the first nonzero lift is cuspidal.
- Study of the lowest point for which the lift is nonzero ("first occurrence") is of interest, related to periods (Roberts).

The Classical Theta Correspondence: Local Aspects

- Weil explained that theta series are closely tied to a representation, the Weil representation ω_ψ, of the double cover $Mp(2n)$ of the symplectic group $Sp(2n)$. (This depends on a choice of additive character ψ of F.)
- Working over a local field, Howe defined the notion of a reductive dual pair and conjectured that if (G, G') is a reductive dual pair then the Weil representation ω_ψ restricts to a correspondence of local representations.
- Howe's conjecture was proved by Howe for archimedean fields (1989), by Waldspurger (1990) for nonarchimedean fields of odd residue characteristic, and by Gan and Takeda (2014) for arbitrary residue characteristic.
- Google scholar: "Howe correspondence": About 364,000 results.
- Functorial when size is roughly the same (Rallis, Adams, Prasad).
- Local theta towers: Kudla (replace "cuspidal" by "supercuspidal").

Beyond the Weil Representation

Natural Question: What makes the Weil representation so special?

Intuition: Given p-adic groups G, G', both subgroups of a larger group H, and a representation Θ of H, one can study the decomposition of Θ into irreducible representations of $G \times G'$. If π is an irreducible admissible representation of G, then the maximal π-isotypic quotient of Θ is of the form $\pi \otimes \Theta(\pi)$ for some representation $\Theta(\pi)$ of G'. If $\Theta(\pi)$ is generally 'not large', then the initial representation Θ must itself be *small*.

Observation (Kazhdan): The Weil representation is a *minimal representation*, i.e. attached to the minimal nontrivial co-adjoint orbit...the Fourier coefficients for all higher unipotent orbits are zero.

Questions: (i) Local: Can one find other minimal representations? Can one study a generalized Howe correspondence? (ii) Global: Can one find other automorphic minimal representations. If so, is it possible to use them to make a theta correspondence? I.e. if Θ is an automorphic minimal representation, study $\int f(g)\,\theta(g,g')\,dg$ where $\theta \in \Theta$ and f is an automorphic form.

Local Minimal Representations

- Local analogue of a Fourier coefficient: a twisted Jacquet module.
- To each unipotent orbit $\mathcal{O}$ attach a unipotent subgroup $U_{\mathcal{O}}$.
- If $\psi_{\mathcal{O}}$ is a generic character of $U_{\mathcal{O}}$, consider the twisted Jacquet module

$$J_{\mathcal{O},\psi_{\mathcal{O}}}(V) = V/ < \pi(u)v - \psi_{\mathcal{O}}(u)v \mid u \in U_{\mathcal{O}}, v \in V > .$$

- Minimal: if $\mathcal{O}$ is any non-minimal unipotent orbit and $\psi_{\mathcal{O}}$ is a generic character then $J_{\mathcal{O},\psi_{\mathcal{O}}}(V) = 0$.
- Savin, Kazhdan-Savin: studied local minimal representations, constructed them; archimedean work of Gross-Wallach, Brylinski-Kostant.

Automorphic Minimal Representations

- Basic observation: The Jacobi theta function

$$\sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$$

  is a residue of the half-integral weight Eisenstein series
- More generally, one can construct the theta series obtained from the Weil representation by *residues of Eisenstein series*, and this is a special case of the Siegel-Weil formula (Ikeda, 1994, 1996).
- This suggests using residues of other Borel Eisenstein series to contruct automorphic "theta functions" that may be minimal representations. Note that there is no Weil representation that can be used to construct the functions in these representation spaces.
- Using residues of Borel Eisenstein series, Ginzburg, Rallis and Soudry (1997, IJM) constructed automorphic minimal representations for split, simply laced groups over number fields.

Exceptional Minimal Theta Correspondences (Local and Global)

Exceptional groups provide other examples of dual pairs, and the minimal representation can be used to get a theta correspondence.

- For example, using E_7: Magaard-Savin (1997), J.-S. Li (1997), Gross-Savin (1998): related to a project to construct motives of rank 7 and weight 0 over $\mathbb{Q}$ with Galois group of type G_2.
- Other examples: Weissman (2006); Huang-Pandzic-Savin (1996), Loke-Savin (2019)
- Ginzburg-Rallis-Soudry (1997): automorphic theta tower involving G_2, using the automorphic minimal representation of GE_7. Further work by Ginzburg-Jiang (IJM, 2001).
- Ginzburg (2019): Using automorphic theta on double cover of F_4.

Automorphic Forms on (Higher) Metaplectic Covers

We will encounter other covers throughout this talk.

- Fix $r > 1$ and let F be a local field or a number field containing a full set μ_r of r-th roots of unity. Let $\underline{G}$ be a linear algebraic group over F.
- Suppose F is global. An "automorphic form" on a covering group of G is a function with the automorphicity property

$$f(\gamma g) = \chi(\gamma)\, f(g)$$

  where χ is a homomorphism of a congruence subgroup $\Gamma \subseteq \underline{G}(\mathcal{O}_F)$ *whose kernel is not a congruence subgroup*. Prototype: The Kubota homomorphism $\kappa\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \left(\frac{c}{d}\right)_r$.
- To write these adelically, let $G = \underline{G}(F)$ (local) or $G = \underline{G}(\mathbb{A})$ (global).
- We write $G^{(r)}$ for a topological central extension of G by μ_r:

$$1 \longrightarrow \mu_r \longrightarrow G^{(r)} \longrightarrow G \longrightarrow 1.$$

- There is a long history here. Some recent references: Gan-Gao-Weissman (2018), Kaplan (2019).

A Non-Minimal Theta Correspondence, I

- Vogan: There is no minimal representation (automorphic or local) for a split odd orthogonal group SO_{2n+1} or any cover when $2n+1 \geq 9$.
- Bump-F-Ginzburg (2003): One can construct an automorphic representation Θ_{2n+1} on the double cover $SO^{(2)}_{2n+1}$ of split SO_{2n+1} that is "small" but (for $2n+1 \geq 9$) not minimal; for SO_9 attached to one of the next two smallest orbits.
- The automorphic representation Θ_{2n+1} is a residue of an Eisenstein series on $SO^{(2)}_{2n+1}$. It is square-integrable and irreducible.
- The corresponding local representation is the image of an intertwining operator M_{w_0} attached to the long Weyl group element:

$$M_{w_0} : \mathrm{Ind}(\delta^{1/2}\chi_\Theta) \longrightarrow \mathrm{Ind}(\delta^{1/2}\chi_\Theta^{-1}).$$

for a suitable character χ_Θ. It is also the unique irreducible quotient of $\mathrm{Ind}(\delta^{1/2}\chi_\Theta)$.

A Non-Minimal Theta Correspondence, II

- Unipotent orbits: correspond to *partitions* of $2n+1$ such that every even part is repeated an even number of times.

Theorem (BFG)

The unipotent orbit attached to Θ_{2n+1} is

$$\mathcal{O}_{2n+1} = \begin{cases} (2^{2m}1) & \text{if } n = 2m \\ (2^{2m}1^3) & \text{if } n = 2m-1. \end{cases}$$

That is: Θ_{2n+1} has nonzero Fourier coefficients

$$\int_{U(F)\backslash U(\mathbb{A})} \theta(ug)\,\psi_U(u)\,du \qquad U = U_{\mathcal{O}_{2n+1}}$$

attached to this orbit, while all Fourier coefficients for higher or incomparable orbits vanish.

- Local results.

A Non-Minimal Theta Correspondence, III

- Even though Θ_{2n+1} is not minimal, one can use it to construct a theta correspondence (BFG, 2006).
- For any two natural numbers $2k+1$ and $2m$ embed the orthogonal groups SO_{2k+1} and SO_{2m} in $\mathrm{SO}_{2k+2m+1}$ as follows:

$$(h,g) \hookrightarrow \begin{pmatrix} a & 0 & b \\ 0 & g & 0 \\ c & 0 & d \end{pmatrix}, \qquad g \in \mathrm{SO}_{2k+1}, h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}_{2m}.$$

- Let π denote an irreducible cuspidal genuine automorphic representation of $SO^{(2)}_{2k+1}(\mathbb{A})$. Then we studied

$$\tilde{f}(h) = \int_{SO_{2k+1}(F)\backslash SO^{(2)}_{2k+1}(\mathbb{A})} \varphi_\pi(g)\, \overline{\theta}_{2k+2m+1}(h,g)\, dg.$$

A Non-Minimal Theta Correspondence, III

- For the n-fold metaplectic cover $SO_k^{(n)}$ of a split special orthogonal group, over a field containing the $2n$-th roots of unity, we proposed:

$$(\widetilde{SO}_{2k+1}^{(n)})^\vee \cong \begin{cases} Sp_{2k}(\mathbb{C}) & \text{if } n \text{ is odd;} \\ SO_{2k+1}(\mathbb{C}) & \text{if } n \text{ is even} \end{cases}$$

  while $(\widetilde{SO}_{2k}^{(n)})^\vee \cong SO_{2k}(\mathbb{C})$ regardless of the parity of n.
- We showed that correspondence from $SO_k^{(2)}$ to $SO_{k+1}^{(2)}$ is functorial (in this sense) on unramified principal series.
- Yusheng Lei (in progress) is studying how the wave front set interacts with this non-minimal theta correspondence.

Another Non-Minimal Theta Lift: S. Leslie's Theta Lifting

Spencer Leslie (2019): Looked at the automorphic theta representation Θ_{2m} on the 4-fold cover $Sp_{2m}^{(4)}(\mathbb{A})$.

- He showed that this representation is attached to the unipotent orbit (2^m).
- He used Θ_{2m} as an integral kernel, to get a lift from $Sp_{2k}^{(4)}(\mathbb{A})$ to $Sp_{2(k+a)}^{(4)}(\mathbb{A})$ (with $m = 2k + a$).
- He showed that there is a tower property, and that the first nonzero lift of a genuine cuspidal representation is a CAP representation.
- He studied the local correspondence for unramified principal series.

Extending the Classical Theta Correspondence

- Work of Friedberg and Ginzburg (arXiv:2006.09305) (2020).
- Extends the classical theta correspondence to all odd degree covers of orthogonal groups and corresponding covers of symplectic groups.
- Suppose that r is odd. Then, both locally and globally, we replace the dual pair

$$(SO_{2a}, Sp_{2l}) \quad \text{by} \quad (SO_{2a}^{(r)}, Sp_{2l}^{(r)})$$

and the dual pair

$$(SO_{2a+1}, Sp_{2l}^{(2)}) \quad \text{by} \quad (SO_{2a+1}^{(r)}, Sp_{2l}^{(2r)}).$$

The Structure Behind Our Construction

Our structure is this:

1. G_1, G_2 groups, $Sp(W_1)$, $Sp(W_2)$ symplectic groups with monomorphisms

$$\iota_1 : G_1 \times G_2 \to Sp(W_1), \quad \iota_2 : G_1 \times G_2 \to Sp(W_2),$$

   such that

   1. via the map ι_1, (G_1, G_2) is a reductive dual pair in $Sp(W_1)$
   2. the images under ι_2 of $G_1 \times 1$ and $1 \times G_2$ in $Sp(W_2)$ commute, though they need not be mutual centralizers.

2. A unipotent group $U \subset Sp(W_2)$ which is normalized by $\iota_2(G_1, G_2)$
3. A character $\psi_U : U \to \mathbb{C}$ such that $\iota_2(G_1, G_2)$, acting by conjugation, stabilizes ψ_U, and the index of $\iota_2(G_1, G_2)$ in the full stabilizer of ψ_U is finite.
4. A homomorphism $\ell : U \to \mathcal{H}(W_1)$, where $\mathcal{H}(W_1)$ denotes the Heisenberg group attached to W_1.

The Structure Behind Our Construction, II

- Let ω_ψ denote the Weil representation of $\mathcal{H}(W_1) \rtimes \widetilde{Sp}(W_1)$ (with additive character ψ).
- Let Θ_2 be a small representation of $Sp(W_2)$.
- Then we form the representation $\omega_\psi \otimes \Theta_2$ of $(\mathcal{H}(W_1) \rtimes \widetilde{Sp}(W_1)) \times Sp(W_2)$.
- Given this information, we propose that (sometimes) there should be a generalization of the Howe and theta correspondences. There are local and global aspects.

The Resulting Correspondences

- Suppose we work locally. *Replace ω_ψ by the twisted Jacquet module $J_{U,\psi_U}(\omega_\psi \otimes \Theta_2)$ defined as follows.* We write (V_1, ω_ψ) and (V_2, Θ_2) as the representations, then $J_{U,\psi_U}(\omega_\psi \otimes \Theta_2) = (V_1 \otimes V_2)/W$ with

$$W =< \omega_\psi(\ell(u))v_1 \otimes \Theta_2(u)v_2 - v_1 \otimes \psi_U(u)v_2 \mid v_1 \in V_1, v_2 \in V_2, \\ u \in U > .$$

  This is naturally a module for $G_1 \times G_2$ (or a cover), where the action on the first factor is via ι_1 and on the second factor via ι_2. Then restrict this Jacquet module to $G_1 \times G_2$. We propose that this should, in certain cases, give a correspondence.
- For the global construction, we replace the Jacquet module by an integral over $[U]$ against the character ψ_U, and restrict to the embedded image of $G_1 \times G_2$ to make an integral kernel.

Specifics of Our Construction, I

Fix $r > 1$ odd, $n \geq 1$, $k \geq 2$. Set $r_1 = (r-1)/2$. Let:

1. $G_1 = SO_k$, $G_2 = Sp_{2n}$ (G_1 is split in our paper, but this is not essential).
2. $\iota_1 : SO_k \times Sp_{2n} \to Sp_{2nk}$ denote the usual tensor product embedding.
3. $\iota_2 : SO_k \times Sp_{2n} \to Sp_{2n+k(r-1)}$ be the embedding given by $\iota_2(h, g) = \mathrm{diag}(h, \ldots, h, g, h^*, \ldots, h^*)$, where h, h^* each appear r_1 times and h^* is determined so that the matrix is symplectic.

Specifics of Our Construction, II

4. $U = U_{k,r_1,n}$ where $U_{k,r_1,n}$ is the unipotent radical of the parabolic subgroup with Levi $GL_k \times \ldots \times GL_k \times Sp_{2n}$ where GL_k appears r_1 times.
5. $\psi_U(u) = \psi(\mathrm{Tr}(X_1 + \cdots + X_{r_1-1}))$ where the X_i are the $k \times k$ blocks above the diagonal. If $r = 7$:

$$\begin{pmatrix} I_k & X_1 & * & * & * & * & * \\ & I_k & X_2 & * & * & * & * \\ & & I_k & * & * & * & * \\ & & & I_{2n} & * & * & * \\ & & & & I_k & X_2^* & * \\ & & & & & I_k & X_1^* \\ & & & & & & I_k \end{pmatrix}$$

Note: this corresponds to the unipotent orbit $((r-1)^k 1^{2n})$.

Specifics of Our Construction, III

6. Let $U_{k,2n}$ be the subgroup of U of all matrices of the form

$$\begin{pmatrix} I_{k(r_1-1)} & & & & \\ & I_k & Y & Z & \\ & & I_{2n} & Y^* & \\ & & & I_k & \\ & & & & I_{k(r_1-1)} \end{pmatrix}, \qquad Y \in \mathrm{Mat}_{k\times 2n}.$$

Then the group $U_{k,2n}$ is isomorphic to a Heisenberg group. The map ℓ is obtained by projecting from U to $U_{k,2n}$ and using this isomorphism.

Specifics of Our Construction, IV

- Let F be a number field containing the r-th roots of unity, and $\mathbb{A}$ its ring of adeles.
- Then we use the representation $\Theta^{(r)}_{2n+k(r-1)}$ on the r-fold cover of $Sp_{2n+k(r-1)}(\mathbb{A})$ as the "small representation" Θ_2 of the general setup. Once again this is not a minimal representation, but (as we shall explain later) it is far from generic.
- The construction involves keeping track of covers and projections from one cover to another. This is suppressed in this talk. To handle the global covering groups, we follow the adelic set-up of the papers of Takeda and Kaplan.

Global Integral

Fix k and let $\kappa = 1$ if k is even, and $\kappa = 2$ if k is odd. Let $\pi^{(\kappa r)}$ be a genuine irreducible cuspidal automorphic representation of $Sp_{2n}^{(\kappa r)}(\mathbb{A})$.

Definition

The theta lift of $\pi^{(\kappa r)}$ is the representation of $SO_k^{(r)}(\mathbb{A})$ generated by the functions $f(h)$ defined by

$$f(h) = \int_{[Sp_{2n}^{(\kappa r)}]} \int_{[U_{k,r_1,n}]} \overline{\varphi^{(\kappa r)}(g)} \, \theta_{2nk}^{(2),\psi}(\ell(u)\iota_1(h,g)) \, \theta_{2n+k(r-1)}^{(r)}(u\,\iota_2(h,g)) \, \psi_U(u) \, du \, dg.$$

as each of $\varphi^{(\kappa r)}$, $\theta_{2nk}^{(2),\psi}$, $\theta_{2n+k(r-1)}^{(r)}$ varies over the functions in its representation space.

Remark: If $r = 1$, then U is trivial and $\theta^{(r)}$ is trivial. In this case this is the classical theta integral.

Global Integral, II

- Similarly, one can define a lift from genuine cuspidal automorphic representations of $SO_k^{(r)}(\mathbb{A})$ to automorphic representations of $Sp_{2n}^{(\kappa r)}(\mathbb{A})$.

Unipotent Orbits for Theta Representations

Question: For a given group, what is the unipotent orbit attached to the theta representation (a residue of the Borel Eisenstein series)?

- For example, for covers of GL_n, Kazhdan and Patterson studied this question. If the cover is of high enough degree, they showed that the theta representation is generic. Yuanqing Cai (2019), and independently, Savin (local results), gave the general answer: if $n = ra + b$ with $0 \leq b < r$ then the orbit is $(r^a b)$.
- For odd degree covers of symplectic groups, we have the following conjecture. *Conjecture:* Let $r > 1$ be an odd integer. Then there is a single unipotent orbit $\mathcal{O}$ that is maximal with respect to the property that $\Theta_{2l}^{(r)}$ has a non-zero Fourier coefficient with respect to $\mathcal{O}$. If $2l = \alpha r + \beta$ where $0 \leq \beta < r$, then this orbit is given by

$$\mathcal{O}_c(\Theta_{2l}^{(r)}) = \begin{cases} (r^\alpha \beta) & \text{if } \alpha \text{ is even} \\ (r^{\alpha-1}(r-1)(\beta+1)) & \text{if } \alpha \text{ is odd.} \end{cases}$$

Description of Main Theorems

Our main results are:

Theorem (F-Ginzburg, 2020)

1. *Fourier Coefficients, Vanishing: For any r, all Fourier coefficients of $\Theta_{2l}^{(r)}$ are zero for unipotent orbits that are larger than or not comparable with the orbit $\mathcal{O}_c(\Theta_{2l}^{(r)})$.*
2. *Fourier Coefficients, Nonvanishing: Let $l = 0, 1, 2, r-3, r-2, r-1$, and let n denote a non-negative integer; if $l = 0$, suppose that $n \geq 1$. Then the Conjecture holds for the group $Sp_{2(l+nr)}^{(r)}(\mathbb{A})$. In particular, when $r = 3, 5$ the conjecture holds for all l and n.*
3. *Theta tower: The first nonzero occurrence in the theta tower is a cuspidal automorphic representation.*
4. *Local correspondence: For equal rank and for unramified principal series, the local theta map described above is functorial.*

"Functorial"

Functorial means with respect to the connected L-groups:

$$(Sp_{2l}^{(r)})^\vee = \begin{cases} SO_{2l+1}(\mathbb{C}) & \text{if } r \text{ is odd} \\ Sp_{2l}(\mathbb{C}) & \text{if } r \text{ is even} \end{cases}$$

$$(SO_{2a+1}^{(r)})^\vee = \begin{cases} Sp_{2a}(\mathbb{C}) & \text{if } r \text{ is odd} \\ SO_{2a+1}(\mathbb{C}) & \text{if } r \text{ is even.} \end{cases}$$

Dimension Equality

- In the case for which there is functorial transfer, our construction satisfies a dimension equality relating
  - the dimensions of the groups in the integral that describes the transfer map and
  - the dimensions of the representations involved

  (that is, their Gelfand-Kirillov dimensions, in the sense of Ginzburg, IJM, 2006).
- Suppose that π and its theta lift σ are both generic and suppose that $n = [k/2]$. Then, assuming that the above Conjecture holds,

$$\dim(\pi) + \dim(\Theta^{(2)}_{2nk}) + \dim(\Theta^{(r)}_{2n+k(r-1)}) = \dim(Sp_{2n}) + \dim(U_{k,r_1,n}) + \dim(\sigma).$$

  See F-Ginzburg, *Dimensions of automorphic representations, L-functions and liftings*, arXiv:1904.07759 (2019), Section 6, for more information.

Work in Progress

- In a second paper, in preparation, we will show that there is a non-zero theta lift provided the Conjecture is true, and discuss in detail where the first occurrence occurs. We will show that, provided the Conjecture holds for given r, a generic cuspidal genuine automorphic representation of $Sp_{2n}^{(r)}(\mathbb{A})$ always lifts nontrivially to a generic genuine representation on $SO_{2n+r+1}^{(r)}(\mathbb{A})$.
- This general structure should allow one to extend the theta maps of BFG and Leslie to higher covers.

A Broader Paradigm, I

The broader paradigm here is this: *constructions in automorphic forms that work for algebraic groups or their double covers should often extend to higher degree metaplectic covers.* We note three examples:

1. Fan Gao has extended Langlands's work on the constant term of Eisenstein series to covers.
2. The doubling integrals of the authors, Cai and Kaplan may be extended to covers (Kaplan; see also Cai for the unfolding for BD covers).
3. The work here indicates that the classical theta correspondence also generalizes.

Weissman has defined a metaplectic L-group. Our suggestion is that not only the formalism of functoriality but also the integrals that give L-functions or correspondences should often generalize. Of course, doing so must involve new ideas, as is the case here.

A Broader Paradigm, II

A fourth possible example: *Langlands-Shahidi Theory for Covers*.

- It is not straightforward to extend Langlands-Shahidi theory to covers as this theory uses the uniqueness of the Whittaker model, and such uniqueness does not hold, even locally, for most covering groups.
- However, it should be possible to generalize this theory by replacing the Whittaker model for a cuspidal automorphic representation τ on a metaplectic covering group with the Whittaker model for the "generalized Speh representation" attached to τ.
- This generalized Speh representation is a residue of an Eisenstein series and defined only conjecturally at the moment. It is expected to have a *unique Whittaker model* by a generalization of *Suzuki's Conjecture*. (Suzuki's Conjecture concerns covers of GL_n and we expect a similar phenomenon in general.)
- We expect that studying Eisenstein series attached to generalized Speh representations will permit an extension of Langlands-Shahidi theory to metaplectic covering groups.

Thank You For Listening

And...looking forward to seeing everyone in Toronto!