Regularity of the free boundary for the two phase Bernoulli problem

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NYU | COURANT

The first meeting with Alessio...

From Ischia 2010...

GNAMPA-ERC SUMMER SCHOOL

13 jun 2010 - 18 jun 2010 [open in google calendar]

created by pratelli on 14 Dec 2009
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Lo GNAMPA, in collaborazione con l'ERC, sta organizzando una scuola estiva di Calcolo delle Variazioni dal 13 al 18 giugno 2010 ad Ischia, presso l'Hotel Continental Terme.

I corsi saranno tenuti da:

Luis Caffarelli (Univ. di Austin): *Regularity theory for quasilinear and fully nonlinear equations*

Vicent Caselles (Univ. Pompeu Fabra): *The total variation model in image processing*

Neil Trudinger (Australian National Univ.): *Title: tba*

Cedric Villani (Ecole Norm. Sup. Lyon): *Smooth and nonsmooth geometrical aspects of optimal transport*

E' previsto un supporto finanziario per i giovani (dottorandi, post- doc e ricercatori) che ne faranno richiesta fino ad un limite massimo di 60 partecipanti. Coloro che intendono partecipare alla scuola (ed eventualmente richiedere il supporto finanziario) sono pregati di iscriversi collegandosi al sito dell'evento.

Prenotazioni entro il 15 Aprile 2010.

Organizzazione Scientifica: Luigi Ambrosio, Nicola Fusco

Organizzazione Locale: Marco Cicalese, Nicola Fusco, Chiara Leone, Anna Verde

Working with Alessio...

...Austin 2011...

Working with Alessio...

...Oberwolfach 2011...

Working with Alessio...

...working hard...

..to nowadays!

The Bernoulli Free Boundary Problem

Let $\lambda_0, \lambda_+, \lambda_- \geq 0$ be given and for $D \subset \mathbb{R}^d$ let us consider

$$J(u, D) = \int_D |\nabla u|^2 + \lambda_+|\{u > 0\}| + \lambda_-|\{u < 0\}| + \lambda_0|\{u = 0\}|.$$

and the minimization problem

(TPBP) $$\min_{u|_{\partial D} = g} J(u, D).$$

where g is a given function.

The Bernoulli Free Boundary Problem: some remarks

A few simple properties.

- Minimizers are easily seen to exist.
- Uniqueness in general fails.
- A minimizers would like to be harmonic where it is $\neq 0$, but the functional might penalize to be always non zero and/or might impose a "balance" between the negative and positive phase

The Bernoulli Free Boundary Problem: some remarks

When $\lambda_0, \lambda_- = 0$ *and* $g \geq 0$, the problem reduces to the *one phase free boundary problem*:

(OPBP)
$$\min_{u=g,\, u\geq 0} \widehat{J}(u, D)$$
$$\widehat{J}(u, D) := \int_D |\nabla u|^2 + \lambda_+ |\{u > 0\}|$$

Motivations

- These problems have been introduced in the 80's by Alt-Caffarelli (OPBP) and by Alt-Caffarelli-Friedmann (TPBP) motivated by some problems in flows with jets and cavities.

  Since then they have been the model problems for a huge class of free boundary problems.

- More recently these types of problems turned out to have applications in the study of shape optimization problems.

Shape Optimization Problems

Let us consider the following minimization problem:

$$\min_{U \subset D} \operatorname{Cap}(U, D) - \lambda |U|$$

where

$$\operatorname{Cap}(U, D) = \min \Big\{ \int_D |\nabla u|^2 \quad u \in W_0^{1,2}(D), u = 1 \text{ on } U \Big\}$$

is the Newtonian capacity of U relative to D. The problem is equivalent to

$$\min_{v \in W_0^{1,2}(D)} \int_D |\nabla v|^2 - \lambda |\{v = 1\}|$$

$$= \min_{v \in W_0^{1,2}(D)} \int_D |\nabla v|^2 + \lambda |\{0 < v < 1\}| - \lambda |D|.$$

$u = 1 - v$ solves a one phase problem.

Shape Optimization Problems

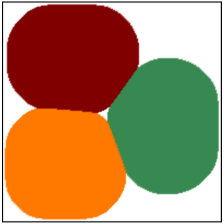
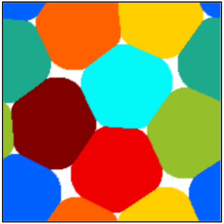
Let us consider the following *minimal partition problem*:

$$\min\Big\{\sum_i \lambda(D_i) + m_i|D_i| \quad D_i \subset D, \quad D_i \cap D_j = \emptyset \text{ if } i \neq j\Big\}.$$

Here $\lambda(D_i)$ is the first eigenvalue of the Dirichlet Laplacian on D_i, i.e.

$$\lambda(D_i) = \inf\left\{ \frac{\int_{D_i} |\nabla u|^2}{\int_{D_i} u^2} : u \in W_0^{1,2}(D_i)\right\}.$$

How minimizers look like?

One can show (Spolaor-Trey-Velichkov):

- There are no *triple points* $\partial D_i \cap \partial D_j \cap \partial D_k = \emptyset$.
- If u_i, u_j are the first (positive) eigenfunctions of D_i, D_j then $v = u_i - u_j$ is a (local) minimizer of

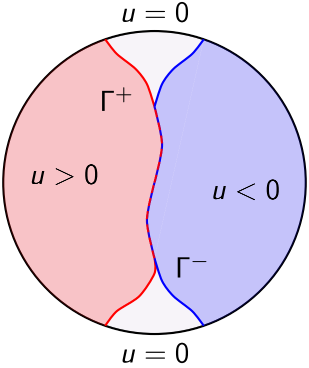
$$\int |\nabla v|^2 + m_i|\{v > 0\}| + m_j|\{v < 0\}| + \text{H.O.T.}$$

Back to the Bernoulli free boundary problem

We are interested in the regularity of u and of the free boundary:

$$\Gamma = \Gamma^+ \cup \Gamma^-$$

$$\Gamma^+ = \partial\{u > 0\} \qquad \Gamma^- = \partial\{u < 0\}.$$



Known results

- u is Lipschitz, Alt-Caffarelli (one phase), Alt-Caffarelli-Friedmann (two-phase).
- If u is a solution of the *one-phase* problem, then Γ^+ is smooth outside a (relatively) closed set Σ_+ with $\dim_{\mathcal{H}} \le d - 5$ (Alt-Caffarelli, Weiss, Jerison-Savin, a recent new proof from De Silva).
- There is a minimizer in dimension $d = 7$ with a point singularity (De Silva-Jerison).
- If u is a solution of the *two phase* problem and $\lambda_0 \ge \min\{\lambda_+, \lambda_-\}$, then $\Gamma^+ = \Gamma^- = \Gamma$ is smooth. (Alt-Caffarelli-Friedmann, Caffarelli, De Silva-Ferrari-Salsa).

The case $\lambda_0 \geq \min\{\lambda_+, \lambda_-\}$

If $\lambda_- \leq \lambda_0$, let v be the harmonic function which is equal to u^- on $\partial(D \setminus \{u > 0\})$. Then

$$w = u^+ - v$$

satisfies

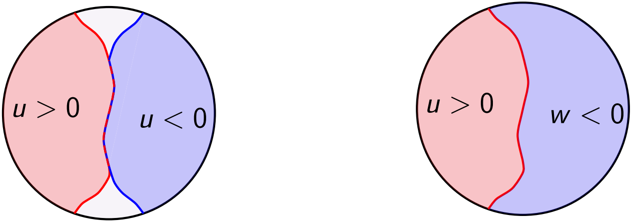
$$J(w, D) \leq J(u, D).$$

since

$$\lambda_- |\{w < 0\}| \leq \lambda_- |\{u < 0\}| + \lambda_0 |\{u = 0\}|$$

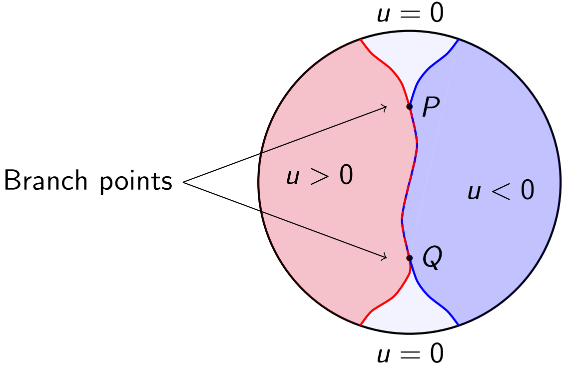
and

$$\int |\nabla v|^2 \leq \int |\nabla u^-|^2$$



The case $\lambda_0 < \min\{\lambda_+, \lambda_-\}$

When $\lambda_0 < \min\{\lambda_+, \lambda_-\}$ the three phases may co-exist and *branch points* might appear.



Main result

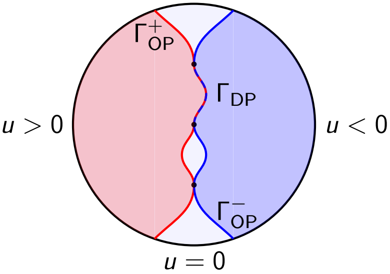
Theorem D.-Spolaor-Velichkov '19 (Spolaor-Velichkov'16 for $d = 2$)

Let u be a local minimizer of J. Let us define

$$\Gamma^{\pm} = \partial\{\pm u > 0\} \qquad \Gamma_{DP} = \Gamma^{+} \cap \Gamma^{-} \qquad \Gamma^{\pm}_{OP} = \Gamma^{\pm} \setminus \Gamma_{DP},$$

Then

- $\Gamma^{\pm}$ *are* $C^{1,\alpha}$ *manifolds outside relatively closed set* $\Sigma^{\pm}$ *with* $\dim_{\mathcal{H}}(\Sigma^{\pm}) \leq d - 5$.
- $\Gamma_{DP} \cap \Sigma^{\pm} = \emptyset$. *In particular* Γ_{DP} *is a closed subset of a* $C^{1,\alpha}$ *graph.*



Steps in the proof

As it is customary in Geometric Measure Theory, the above result is based on two steps:

- Blow up analysis.
- ε-regularity theorem.

Optimality conditions

Before detailing the proof, let us start by deriving the optimality conditions for minimizers.

The first (trivial) one, one is that u is harmonic where $\neq 0$ (which is open)

$$\Delta u = 0 \qquad \text{on } \{u \neq 0\}$$

What are the optimality conditions on the free boundary?

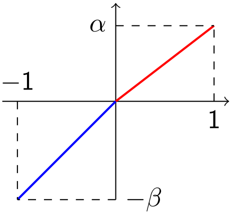
They can be formally obtained by performing inner variations

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J(u_\varepsilon) = 0 \qquad u_\varepsilon(x) = u(x + \varepsilon X(x)) \quad X \in C_c(D; \mathbb{R}^d)$$

Optimality condition

Let us assume that u is one dimensional:

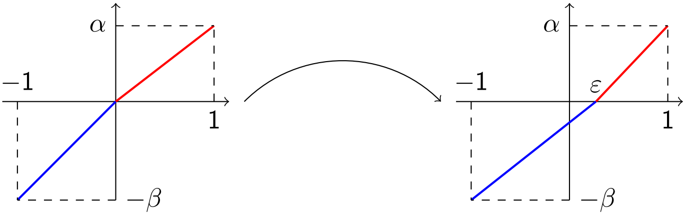
$$u = \alpha x_+ - \beta x_-$$



Optimality condition

Let us assume that u is one dimensional:

$$u = \alpha x_+ - \beta x_- \qquad u_\varepsilon = \frac{\alpha}{1-\varepsilon}(x-\varepsilon)_+ - \frac{\beta}{1+\varepsilon}(x-\varepsilon)_-$$

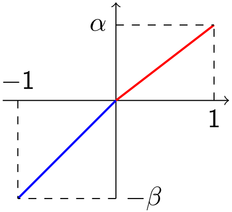


$$0 \leq J(u_\varepsilon) - J(u) = (\alpha^2 - \beta^2)\varepsilon - (\lambda_+ - \lambda_-)\varepsilon + o(\varepsilon)$$

Optimality condition

Moreover

$$u = \alpha x_+ - \beta x_-$$



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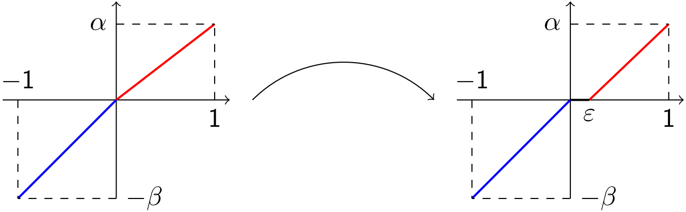
$$u = \alpha x_+ - \beta x_- \qquad\qquad u_\varepsilon = \frac{\alpha}{1-\varepsilon}(x-\varepsilon)_+ - \beta x_-$$



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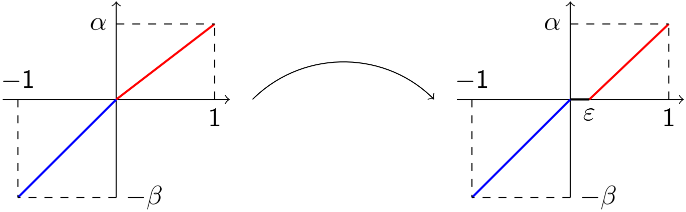

$$\begin{aligned} 0 \leq J(u_\varepsilon) - J(u) &= \frac{\alpha^2}{(1-\varepsilon)} - \alpha^2 - (\lambda_+ - \lambda_0)\varepsilon \\ &= \alpha^2\varepsilon - (\lambda_+ - \lambda_0)\varepsilon + o(\varepsilon) \end{aligned}$$

Optimality conditions

We get the following problem

$$\begin{cases} \Delta u = 0 & \text{on } \{u \neq 0\} \\ |\nabla u^{\pm}|^2 = \lambda_{\pm} - \lambda_0 & \text{on } \Gamma^{\pm}_{\mathrm{OP}} \\ |\nabla u^{+}|^2 - |\nabla u^{-}|^2 = \lambda_{+} - \lambda_{-} & \text{on } \Gamma_{\mathrm{DP}} \\ |\nabla u^{\pm}|^2 \geq \lambda_{\pm} - \lambda_0 & \text{on } \Gamma^{\pm} \end{cases}$$

Blow up analysis

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Let $x_0 \in \Gamma$ and $r > 0$. Let

$$u_{x_0,r}(x) = \frac{u(x_0 + rx)}{r} \qquad (u(x_0) = 0).$$

Then $\{u_{x_0,r}\}_{r>0}$ is pre-compact in C^0 and every limit point is *one-homogeneous* (Weiss Monotonicity Formula).

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If $x_0 \in \Gamma$ is regular it is easy to see that there is a unique limit v_{x_0} and

$$v_{x_0} = \begin{cases} \pm\sqrt{\lambda_\pm - \lambda_0}(x \cdot \boldsymbol{e}_{x_0})_\pm & \text{if } x_0 \in \Gamma_{\mathrm{OP}}^{\pm} \\ \alpha_+(x \cdot \boldsymbol{e}_{x_0})_+ - \alpha_-(x \cdot \boldsymbol{e}_{x_0})_- & \text{if } x_0 \in \Gamma_{\mathrm{DP}} \\ \qquad \alpha_\pm \geq \sqrt{\lambda_\pm - \lambda_0}, \quad \alpha_+^2 - \alpha_-^2 = \lambda_+ - \lambda_- & \end{cases}$$

where $\boldsymbol{e}_{x_0}$ is the normal to Γ at x_0.

Regular points

We are going to call a point *regular* if $\{u_{x_0,r}\}$ admits *one* limit point of the above form (for some $\boldsymbol{e}$).

1. One can prove that the complement of regular point has small dimension (Federer dimension reduction) and it does not intersect the two phase free boundary (these are $\Sigma^{\pm}$).
2. The difficult part consists in proving that if x_0 is regular the Γ has the desired structure in a neighborhood, in particular *all* blow-up coincide. (ε-regularity theory).
3. The ε regularity theory was known at one phase points (Alt-Caffarelli, De Silva) and at points which are at the interior of the two phase free boundary (Caffarelli, De Silva-Ferrari-Salsa)
4. The new step is to understand what happens at *branch points* and to put everything together.

The ε-regularity theorem at one phase point

Let us show De Silva's proof at *one-phase* points ($\lambda_+ = 1, \lambda_-, \lambda_0 = 0$, $\boldsymbol{e} = \boldsymbol{e}_1$).
Assume that in B_1

$$u^+ \approx (x_1)_+ \qquad u^+ = x_1 + \varepsilon v_\varepsilon \quad \text{on } \{u > 0\} \qquad \varepsilon := \|u^+ - x_1\|_{L^\infty(\{u>0\}\cap B_1)}$$

What are the equation satisfied by v_ε?

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What are the equation satisfied by v_ε?

$$\Delta v_\varepsilon = 0 \qquad \text{on } \{u > 0\} \approx B_1^+.$$

moreover

$$1 = |\nabla u^+|^2 = 1 + \varepsilon \partial_1 v_\varepsilon + o(\varepsilon) \qquad \text{on } \partial\{u > 0\}$$

i.e.

$$\partial_1 v_\varepsilon \approx 0 \qquad \text{on } \partial\{u > 0\} \approx \{x_1 = 0\}$$

The ε-regularity theorem at one phase point

In other words v_ε is almost a solution of a Neumann problem

(NP) $$\begin{cases} \Delta v = 0 & \text{on } B_1^+ \\ \partial_1 v = 0 & \text{on } \{x_1 = 0\} \cap B_1 \end{cases}$$

The C^2 regularity theory for the (NP) allows to show the existence of

$$\mathbb{S}^{d-1} \ni \boldsymbol{e} = \boldsymbol{e}_1 + \varepsilon \nabla v(0) + O(\varepsilon^2) \qquad \big(\boldsymbol{e}_1 \perp \nabla v(0)\big)$$

such that for $\rho, \delta \ll 1$

$$\|u^+ - (x \cdot \boldsymbol{e})_+\|_{L^\infty(\{u>0\}\cap B_\rho)} \le \rho^{2-\delta} \|u^+ - (x_1)_+\|_{L^\infty(\{u>0\}\cap B_1)}.$$

What happens at branch points?

Assume $\lambda_\pm = 1$, $\lambda_0 = 0$. At branch points

$$u \approx (x_1)_+ - (x_1)_- + \varepsilon v_\varepsilon^+ + \varepsilon v_\varepsilon^-$$

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The functions $v_\varepsilon^\pm$ are almost solutions of a thin two membrane problem (this was first observed by Andersson-Shahgholian-Weiss).

$$\begin{cases} \Delta u = 0 & \text{on } \{u \neq 0\} \\ |\nabla u^\pm|^2 = 1 & \text{on } \Gamma^\pm_{\mathrm{OP}} \\ |\nabla u^+|^2 = |\nabla u^-|^2 & \text{on } \Gamma_{\mathrm{DP}} \\ |\nabla u^\pm|^2 \geq 1 & \text{on } \Gamma^\pm \end{cases} \Rightarrow \begin{cases} \Delta v^\pm = 0 & \text{on } B_1^\pm \\ \partial_1 v^\pm = 0 & \text{on } \{v^+ \neq v^-\} \cap \{x_1 = 0\} \\ \partial_1 v^+ = \partial_1 v^- & \text{on } \{v^+ = v^-\} \cap \{x_1 = 0\} \\ \partial_1 v^\pm \geq 0 & \text{on } \{x_1 = 0\} \end{cases}$$

$C^{1,\frac{1}{2}}$ regularity for the two membrane problem would allow to conclude (same caveat).

The ε-regularity theorem at one phase point: compactness

The key point to make the above proofs rigorous is *compactness* of $v_\varepsilon^\pm$.

The ε-regularity theorem at one phase point: compactness

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A good topology is C^0 (solutions will be intended in the viscosity sense) which is the topology where the sequences are bounded. Some a-priori regularity theory is needed (De Silva: adapt Savin's "Partial Harnack inequality").

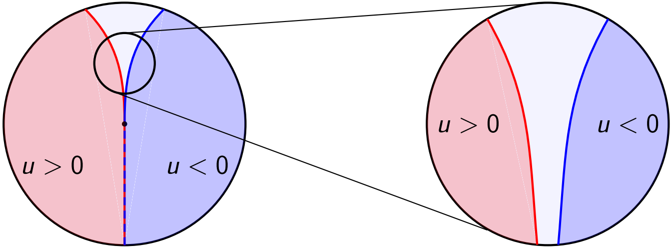
Compactness at branch points

In order to prove compactness one does not only to deal with the case where

$$u \approx (x_1)_+ - (x_1)_- \qquad \text{but also} \qquad u \approx (x_1 + \delta_1)_+ - (x_1 + \delta_1)_-$$

with $\delta_1, \delta_2 \ll 1$. This is the behavior close to branch points.

Indeed this is the local picture close a branch point:



Congratulations Alessio for this well deserved achievement...

...and let us cheer to the next ones!

THANK YOU
FOR YOUR ATTENTION!