The singular set in the Stefan problem

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Free boundary problems

- Any PDE problem that exhibits apriori unknown (free) interfaces or boundaries
- They appear in Physics, Industry, Finance, Biology, and other areas
  - Most classical example:

    Stefan problem (1831)

    It describes the melting of ice.

  - If $\theta(t,x)$ denotes the temperature,

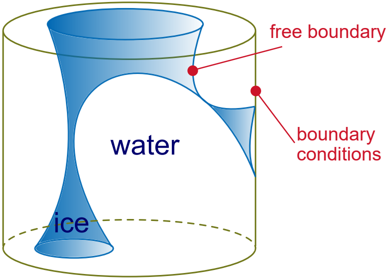
$$\theta_t = \Delta\theta \quad \text{in} \quad \{\theta > 0\}$$

  - Free boundary determined by:

$$|\nabla_x \theta|^2 = \theta_t \quad \text{on} \quad \partial\{\theta > 0\}$$

  - $u := \int_0^t \theta$ solves: $u \geq 0, \quad u_t \geq 0,$

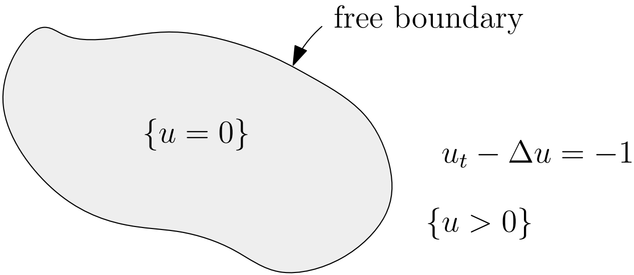
$$\boxed{u_t - \Delta u = -\chi_{\{u>0\}}}$$



$$\boxed{\begin{aligned} u &\geq 0 && \text{in} && \Omega, \\ u_t - \Delta u &= -1 && \text{in} && \{u > 0\} \\ \nabla u &= 0 && \text{on} && \partial\{u > 0\}. \end{aligned}} \quad \longleftrightarrow \quad \boxed{\begin{aligned} u \geq 0 \quad &\text{in } \Omega \\ u_t - \Delta u = -\chi_{\{u>0\}} \quad &\text{in } \Omega \end{aligned}}$$

Unknowns: solution u & the contact set $\{u = 0\}$

The free boundary (FB) is the boundary $\partial\{u > 0\}$



Probability: Optimal stopping $\rightsquigarrow$ Stefan problem

- Let X_t be a Brownian motion in $\mathbb{R}^n$, φ a payoff function.

  We can stop X_t at any time $\tau \in [0, T]$, and we get a payoff $\varphi(X_\tau)$.

- Question: We want to maximize the payoff.

  Should we stop if we are at $x \in \mathbb{R}^n$ at time $t \in [0, T)$?

  We define the value function $$\boxed{v(x,t) = \max_{\text{all choices of } \tau} \mathbb{E}\big[\varphi(X_\tau)\big]}$$

- Then $\boxed{u := v - \varphi}$ solves a **Stefan problem** in $\mathbb{R}^n$!

- The "exercise region" is $\{v = \varphi\}$ (that is, the "ice" $\{u = 0\}$).

- These models are used in Mathematical Finance.

  A typical example is the pricing of American options.

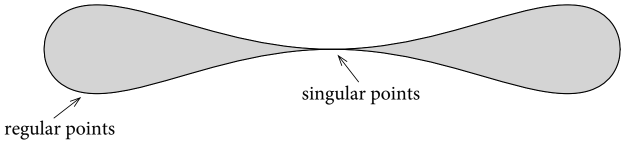
The Stefan problem

Fundamental question:

Is the Free Boundary smooth?

- First results (1960's & 1970's): Solutions u are $C^{1,1}_x \cap C^1_t$, and this is optimal.
- Kinderlehrer-Nirenberg (1977): If the FB is C^1, then it is C^∞
- Caffarelli (Acta Math. 1977): The FB is C^1 (and thus C^∞),

  possibly outside a certain set of singular points



- Let us look at the proof of this result.

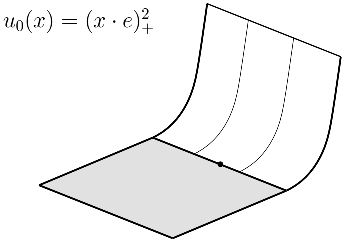
To study the regularity of the FB, one considers blow-ups

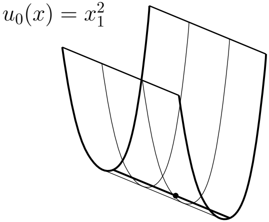
$$u_r(x) := \frac{u(rx, r^2 t)}{r^2} \longrightarrow u_0(x, t)$$

The key difficulty is to **classify blow-ups**:

$$\text{regular point} \quad \Longrightarrow \quad u_0(x) = (x \cdot e)_+^2 \quad \text{(1D solution)}$$

$$\text{singular point} \quad \Longrightarrow \quad u_0(x) = x^T A x \quad \text{(paraboloid)}$$





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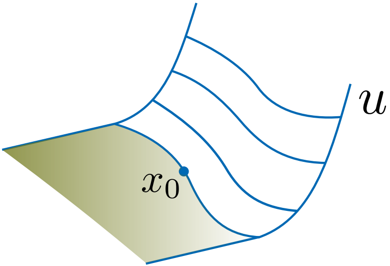
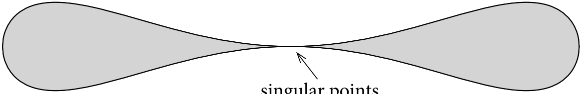
Finally, once the blow-ups are classified, we transfer the information from u_0 to u , and prove that the FB is C^1 near regular points.



Singular points

Question: What can one say about singular points?



- Caffarelli'98 & Monneau'00 & Blanchet'06: In **space**, singular points are contained in a $(n-1)$-dimensional C^1 manifold.

  Moreover, if $(0,0)$ is a singular point, we have

  $$u(x,t) = p_2(x) + \boxed{o(|x|^2 + |t|)},$$

  where p_2 is the blow-up.

- In the elliptic setting, several improvements of this result have been obtained by Weiss (1999), Colombo-Spolaor-Velichkov (2017), Figalli-Serra (2017).

Singular points

Question : *What can one say about the size of the singular set?*

- The previous result implies that, for each time t, the singular set is contained in a C^1 manifold of dimension $(n-1)$.
- However, such manifold is only $C^{1/2}$ in time — recall $\boxed{o(|x|^2+|t|)}$.
  This does not even yield that the singular set is $(n-1)$-dimensional in space-time.
- The following question has been open for years:

Question : *Is the singular set $(n-1)$-dimensional in space-time?*

- The most natural way to measure this is in the parabolic distance

$$d_{\mathrm{par}}((x_1,t_1),(x_2,t_2)) := \sqrt{|x_1-x_2|^2+|t_1-t_2|}$$

and the corresponding parabolic Hausdorff dimension $\dim_{\mathrm{par}}(E)$

Singular points: new results

- In a forthcoming work with Figalli and Serra, we establish for the first time:

Theorem (Figalli-R.-Serra, '20)

Let $u(x,t)$ be any solution to Stefan problem, and Σ be the set of singular points. Then,

$$\boxed{\dim_{\mathrm{par}}(\Sigma) \le n-1}$$

where $\dim_{\mathrm{par}}(E)$ denotes the parabolic Hausdorff dimension of a set $E \subset \mathbb{R}^n \times \mathbb{R}$.

- This is sharp, since Σ could be $(n-1)$-dimensional even for a fixed time $\{t = t_0\}$.
- Since the time axis has parabolic dimension 2, our result implies that, in $\mathbb{R}^2$, the free boundary is smooth for *almost every time t*.
- It is then natural to ask: Does the same happen in $\mathbb{R}^3$?

  "How often" do singular points appear?

The Stefan problem in $\mathbb{R}^3$

- In $\mathbb{R}^3$, we establish the following.

Theorem (Figalli-R.-Serra '20)

Let $u(x,t)$ be the solution to the Stefan problem in $\mathbb{R}^3$.

Then, for almost every time t, the free boundary is C^∞ (with no singular points).

Furthermore, if we define $\mathcal{S}$ as the set of "singular times", then

$$\boxed{\dim_{\mathcal{H}}(\mathcal{S}) \leq \frac{1}{2}}$$

- We need a much finer understanding of singular points in order to prove this!
- Is the $\frac{1}{2}$ sharp? We don't know, but it is definitely *critical*.

Dimension of the singular set: ideas of the proofs

- Let us discuss next the proof of:

Theorem (Figalli-R.-Serra, '20)

Let $u(x,t)$ be any solution to Stefan problem, and Σ be the set of singular points. Then,

$$\boxed{\dim_{\mathrm{par}}(\Sigma) \leq n-1}$$

where $\dim_{\mathrm{par}}(E)$ denotes the parabolic Hausdorff dimension of a set $E \subset \mathbb{R}^n \times \mathbb{R}$.

- To prove it, it would suffice to prove, at all singular points,

$$\left|u(x,t) - p_2(x)\right| \leq Cr^3,$$

where $r = \sqrt{|x|^2 + |t|}$. (Previous results only gave $o(r^2)$.)

- Unfortunately, this is not true at all points!

Dimension of the singular set: ideas of the proofs

- We actually need to prove that, if $(0,0) \in \Sigma$, then

$$\Sigma \cap \left(B_r \times \{t \geq Cr^2\}\right) = \varnothing. \qquad (*)$$

- To prove this, the idea is to combine a "cleaning lemma" with a new expansion

$$\left|u(x,t) - p_2(x)\right| \leq Cr^3. \qquad (**)$$

- However, (**) is not true at all singular points! We split Σ as follows:
- Let Σ_m, where $\{p_2 = 0\}$ is m-dimensional. When $m \leq n-2$, the estimate

$$\left|u(x,t) - p_2(x)\right| \leq o(r^2)$$

cannot be improved! But the barrier is then better, so we get (*).

- In Σ_{n-1}, we can prove (**) at "*most*" points.

In the remaining ones, $\Sigma_{n-1}^{<3}$, we need to use carefully their structure.

Dimension of the singular set: ideas of the proofs

- To establish these higher order estimates (**), we study second blow-ups:

$$\frac{(u-p_2)(rx,r^2t)}{\|u-p_2\|_{Q_r}} \longrightarrow q(x,t)$$

- For this, we need a suitable truncated parabolic version of Almgren's monotonicity formula.
- In Σ_m, $m \leq n-2$, we *always* get a quadratic polynomial again!
- In Σ_{n-1}, we can prove that q is cubic at "*most*" points (via dimension reduction).
- However, q is **not a polynomial** as in the elliptic case!

$$q(x,t) = \tfrac{1}{6}|x_n|^3 + t|x_n|$$

- We cannot continue with a next blow-up: Almgren fails for $w := u - p_2 - q$!
- We need completely new ideas if we want to go further.

Singular points: new results

- We can say much more, and actually establish the following higher order result:

Theorem (Figalli-R.-Serra, '20)

Let $u(x,t)$ be any solution to Stefan problem, and Σ be the set of singular points.

Then, there is Σ^*, with

$$\dim_{\rm par}(\Sigma \setminus \Sigma^*) \leq n-2$$

such that Σ^* is contained in a countable union of C^∞ manifolds of dimension $(n-1)$.

- This substantially improves all known results, and it is even better than our results for the elliptic setting!
- Basically, in Σ^* we get a higher order expansion of order ∞

Higher order expansion at singular points

- To establish our result, we need to improve substantially the understanding of singular points:

Theorem (Figalli-R.-Serra, '20)

Let $u(x,t)$ be any solution to the Stefan problem, and Σ be the set of singular points.

Then, there exists a set $\Sigma^* \subset \Sigma$, with $\dim_{\mathrm{par}}(\Sigma \setminus \Sigma^*) \leq n-2$, such that in Σ^* we have

$$u(x,t) = \frac{1}{2}\Big(x\cdot e + \mathrm{V}^+ t + q_+(x,t)\Big)_+^2 + \frac{1}{2}\Big(x\cdot e - \mathrm{V}^- t + q_-(x,t)\Big)_-^2 + o\left(\left(|x| + \sqrt{|t|}\right)^k\right) \qquad (1)$$

for all $k > 0$, and for $t < 0$.

Here, $e \in \mathbb{S}^{n-1}$, $\mathrm{V}^\pm > 0$, and $q_\pm$ are higher order polynomials (satisfying certain compatibility conditions).

Comments

- This gives a much deeper geometric understanding of "most" singularities: $\mathbf{V}^{\pm}$ are the velocities of two fronts that collapse at $t = 0$, while $q_{\pm}$ correspond to curvature terms.
- The dimension $n - 2$ of the set $\Sigma \setminus \Sigma^*$ is sharp!
- The most difficult step is to pass from order r^3 to $r^{3+\alpha}$. This requires a variety of new ideas, combining GMT tools, PDE estimates, dimension reduction... all *without* monotonicity formulas!
- After order $r^{3+\alpha}$ the proof is completely different, and we get then r^k (for all k) with an approximation argument that may have its own interest.

On the size of the singular set

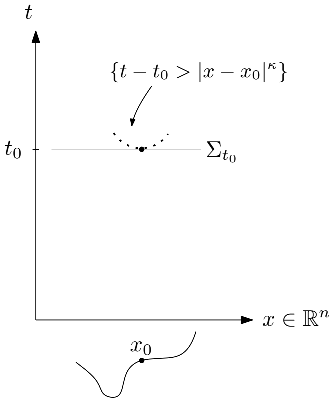
Thanks to such expansions, plus

a "cleaning argument", we get:

- If u has a singular point at (x_0, t_0), then there are no singular points for $u(\cdot, t_0 + r^\kappa)$ in a ball of radius r (for a certain κ).

- In Σ^*, we can take $\kappa \to \infty$, and thus the singular set is C^∞-flat there!

- Thanks to this, the projection of Σ^* on the t-axis has zero Hausdorff dimension.



The Stefan problem in $\mathbb{R}^2$

- Since $\dim_{\rm par}(\Sigma \setminus \Sigma^*) \le n-2$, then in $\mathbb{R}^2$ we deduce the following.

Theorem (Figalli-R.-Serra '20)

Let $u(x,t)$ be the solution to the Stefan problem in $\mathbb{R}^2$.

Let $\mathcal{S}$ be the set of "singular times". Then,

$$\boxed{\dim(\mathcal{S}) = 0}$$

- Proof (2D and 3D): Split Σ into Σ^* and $\Sigma \setminus \Sigma^*$, and apply the previous results.
- Recall: even the regularity for almost every time is new!
- The expansion up to order ∞ is essential here.

Thank you!