Optimal transport in Brownian motion stopping

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Focusing on joint works with
Nassif Ghoussoub (UBC) and **Aaron Zeff Palmer (UBC)**,

and joint work in progress with
Inwon Kim (UCLA).

October, 2020
Fields Medal Symposium, celebrating mathematical work of
Alessio Figalli.

The main works to present

Part I

- **Brownian stopping with fixed target**
  - [Ghoussoub/ K. / Palmer] PDE methods for Skorokhod embeddings. *Calc. Var. PDE (2019)*
  - [Ghoussoub/ K. / Palmer] A solution to the Monge transport problem for Brownian martingales. To appear in *Ann. of Probability*.

Part II

- **Brownian stopping with free target**
  [Inwon Kim/ K.] Work in progress.

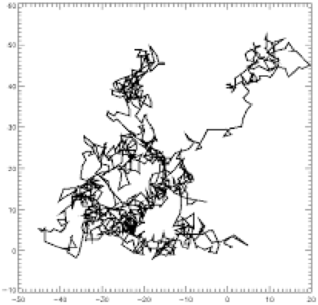
Main point to present:
Optimal transport with Brownian motion stopping has a fundamental connection to free boundary problems of PDEs (the heat equation).

Outline

- Brownian motion, stopping time, Skorokhod problem
- Fixed target
  - Optimal Stopping time (Optimal Skorokhod Problem)/ Connection to Optimal Transport.
  - Randomized stopping time and Kantorovich solution
  - Monge solution, barrier and hitting time
  - Duality/ Dynamic programming
  - Dual attainment
  - Eulerian formulation
- Free target
  - The density constraint optimization problem
  - Monotonicity/ L^1 contraction/BV estimates
  - Saturation
  - Connection to the Stefan problem (a free boundary PDE problem): Freezing / Melting

Brownian motion and stopping time

- **Brownian motion**:



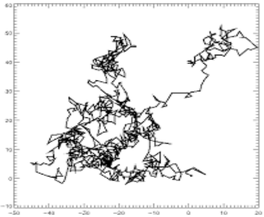
from CRM-physmath

- A **stopping time** τ of Brownian motion is, roughly speaking, a random time, prescribed to satisfy a certain probabilistic condition, at which one stops a particle following the Brownian motion.

Brownian motion and stopping time

[Skorokhod problem in R^n]
For given probability measures μ, ν, does there exist a **stopping time** τ of the Brownian motion such that

$$B_0 \sim \mu \quad \& \quad B_\tau \sim \nu?$$



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Remark:

- For such a stopping time τ to exist (with $\mathbb{E}[\tau] < \infty$), we need
  - μ and ν are in **subharmonic** order, $\mu \prec_{SH} \nu$, i.e. $\int \xi d\mu \leq \int \xi d\nu$, $\forall$ subharmonic $\xi : \mathbb{R}^n \to \mathbb{R}$ ($\Delta \xi \geq 0$).

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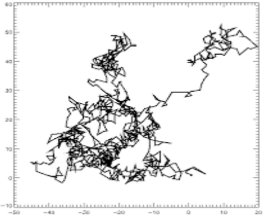

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Skorokod problem

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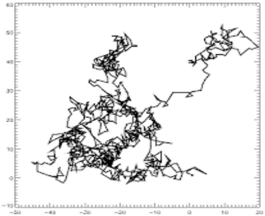


from CRM-physmath

- [Skorokhod] [Root] [Rost] [Azéma/Yor] [Vallois] [Perkins] [Jacka] ...[Obloj]...
- **[Hobson] .. ....**
- **[Beiglböck/Cox/Huesmann '13]**.
  - **Optimal transport** unifies the previous results on Skorokhod problem.
- And many many more people.

Optimal Skorokhod problem

Question: What can we say
about an **optimal** stopping time τ for

$$\mathcal{P}(\mu,\nu) := \inf_{\tau}\{\ \mathcal{C}(\tau) \quad | \quad B_0 \sim \mu \quad \& \quad B_\tau \sim \nu\}?$$

where $\mathcal{C}(\tau) = \mathbb{E}\left[\int_0^\tau L(t, B_t)dt\right]$ or $\mathcal{C}(\tau) = \mathbb{E}\left[|B_0 - B_\tau|\right]$, etc.

- Existence?
- Uniquenss?
- Any extremal structure?
  - Does τ drop mass only in a special type of set?

Optimal transport

Optimal Skorokhod problem is a version of **optimal transport** where the **additional constraint** is given on how mass moves.

- $T(\mu, \nu)$:
  probability measures π on $\mathbb{R}^n \times \mathbb{R}^n$
  with the marginals μ, ν.

Monge-Kantorovich problem:

$$\inf_{\pi \in T(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)\, d\pi(x,y).$$

[Monge]**[Kantorovich]**[Brenier][McCann][Delanoë][Urbas]
[Caffarelli][Evans/Gangbo][Gangbo/McCann][Benamou/Brenier]
[Trudinger/Wang][Ambrosio] [Caffarelli/Feldman/McCann]
[Otto][Otto/Villani]**[Villani]** [Lott/Villani][Sturm]
[Ma/Trudinger/Wang][Loeper] ....**[Figalli]**.....
.....and many more people ....

Martingale optimal transport/Optimal Skorokhod:

- Backhoff, Bayraktar, Beiglböck, Bouchard, Claisse, Cox, Davis, Dolinsky, De March, Galichon, Ghoussoub, Griessler, Guo, Henry-Labordère, Hobson, Hu, Huesmann, Juillet, Kallblad, K., Klimmek, Lim, Neuberger, Nutz, Oblój, Palmer, Penkner, Perkowski, Proemel, Schachermayer, Siorpaes, Soner, Spoida, Stebegg, Tan, Touzi, Zaev, and many more people$\cdots\cdots$.

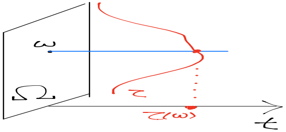
Optimal Skorokhod problem with given μ and ν.

From now on we assume that $\operatorname{supp}\mu, \operatorname{supp}\nu$ are compact in $\mathbb{R}^n$.

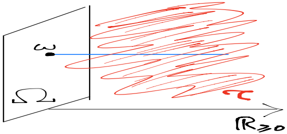
Randomized stopping time

Let $\Omega := C(\mathbb{R}_{\geq 0}; \mathbb{R}^n)$.

Stopping time
is a (certain) measurable **function** τ on the probability space $(\Omega, \mathbb{P}^\mu)$.
($\mathbb{P}^\mu$= the Wiener measure with $B_0 \sim \mu$).



Randomized stopping time [Baxter & Chacon '77, Meyer '78]
is a (certain) probability **measure** τ on the space $\mathbb{R}_{\geq 0} \times \Omega$,
whose marginal on Ω is $\mathbb{P}^\mu$.



A (nonradomized) stopping time gives Dirac mass along each path.

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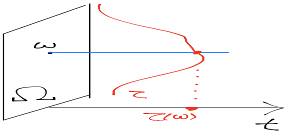

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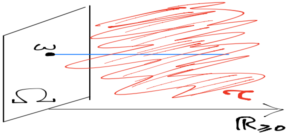

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Optimal Skorokhod problem: Kantorovich solution (a measure-valued solution)

- **[Beiglböck, Cox & Huesmann '13]**
  Randomized stopping times give
  Kantorovich relaxation to optimal Skorokhod problem.
  - **The set of randomized stopping times** from μ to ν is nonempty if $\mu \prec_{SH} \nu$.
  - Space of randomized stopping times is compact: weak* -**compactness** of the space of probability measures.
  - Optimal randomized stopping time exists through lower semi-continuity of the functional $\tau \to \mathcal{C}(\tau)$ over **randomized stopping times**.

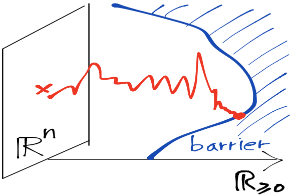
Optimal Skorokhod problem: Monge solution?

- **Question:**
  - When is the optimal Kantorovich solution a Monge solution?
    - In what case, does the optimal randomized stopping time become pure, that is, non-randomized, pure stopping time?
  - Any associated structure?

Optimal Skorokhod problem: Monge solutions (non-randomized stopping)

[Beiglböck, Cox, & Huesmann '13].

- Some **variational** tools in the path space Ω, called monotonicity principle, comparing different paths.
- geometric structures for the cost $\mathbb{E}\left[\int_0^\tau L(t)dt\right]$.
  - The optimal stopping time is unique and given by hitting a certain **barrier in the space-time** $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$



- **Barrier** $R \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$
- The **hitting time** τ^R to R,

$$\tau^R := \inf\{t \geq 0 \mid (t, B_t) \in R\}.$$

Some literature in 1D

Barriers for optimal stopping and obstacle problems for the heat equation:

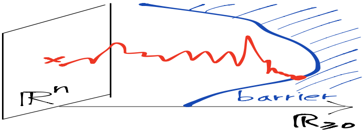
- [McConnell'91]:
- [Cox/Wang '13]
- [Gassiat/Oberhauser/dos Reis '15]
- [DeAngelis,T '18]
- ....

Optimal Skorokhod problem:
Monge solutions (non-randomized stopping)

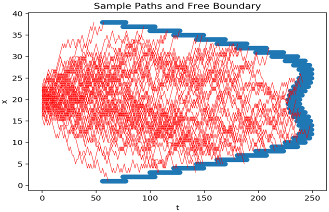
[Ghoussoub, K. & Palmer '18-'19].

- Some **analytical/PDE** tools based on dual formulation.
  - **dual attainment for general dimensions** n.
- geometric structures
  - For $\mathbb{E}\left[\int_0^\tau L(t, B_t)dt\right]$:
    - The optimal sstopping time is uniquely determined by hitting a certain **barrier in the space-time** $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ **given by the optimal dual function**.
  - For $\mathbb{E}\left[|B_0 - B_\tau|\right]$ ($\mathbb{E}\left[d(B_0, B_\tau)\right]$ in Riemannian case):
    - The optimal stopping time is uniquely determined by hitting a certain **barrier in the product space** $\mathbb{R}^n \times \mathbb{R}^n$ **given by the optimal dual function**.

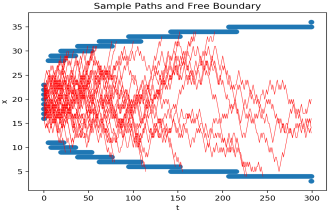
Markovian cost $\mathcal{C}(\tau) = \mathbb{E}\left[\int_0^\tau L(t, B_t)dt\right]$.



Barrier looks like the graph of a function on $\mathbb{R}^n$.



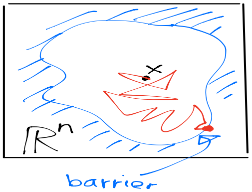
hitting from below
when $t \mapsto L(t, x) \nearrow$
[Root's solution]



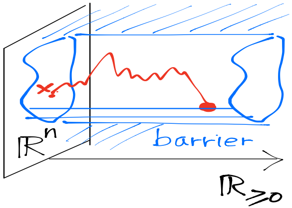
hitting from above
when $t \mapsto L(t, x) \searrow$
[Rost's solution]

Non-Markovian cost $\mathcal{C}(\tau) = \mathbb{E}[|B_0 - B_\tau|]$

Barrier
$R = \{(x, y) | \ y \in R_x\}$
$\subset \mathbb{R}^n \times \mathbb{R}^n$.
The barrier R_x depends on the starting point $x \in R^n$.



In the space time, the barrier R_x (depending on the starting point x) looks like a vertical wall in the space-time.



Tools in [Ghoussoub/ K./ Palmer]:

Duality: $\mathcal{P}(\mu,\nu) = \mathcal{D}(\mu,\nu)$ where

$$\mathcal{D}(\mu,\nu) := \sup_{\psi \in LSC} \Big\{ \int_{\mathbb{R}^d} \psi(z)\nu(dz) - \int_{\mathbb{R}^d} "J_\psi" \mu(dx) \Big\}.$$

Dynamic programming:

- Markovian: $"J_\psi" = J_\psi(0,x)$ where

$$J_\psi(t,y) := \sup_{\tau \in \mathcal{R}} \Big\{ \mathbb{E}\Big[\psi(B_\tau^y) - \int_0^\tau L(t+s, B_s^y)ds \Big] \Big\}.$$

- Non-Markovian: $"J_\psi" = J_\psi(x,x)$ where

$$J_\psi(x,y) := \sup_{\tau \in \mathcal{R}} \Big\{ \mathbb{E}\Big[\psi(B_\tau^y) - c(x, B_\tau^y) \Big] \Big\}.$$

Tools in [Ghoussoub/ K./ Palmer]:

Dynamic programming principle
ψ determines J_ψ that solves (in viscosity sense)

- (Markovian)

$$\min \left\{ \begin{array}{r} J(t,y) - \psi(y) \\ -\frac{\partial}{\partial t} J(t,y) - \frac{1}{2}\Delta J(t,y) + L(t,y) \end{array} \right\} = 0.$$

- (NonMarkovian)

$$\min[J(x,y) - \psi(y) + c(x,y),\ -\Delta_y[J(x,y)] = 0$$

Tools

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Remark: It is nontrivial to find the dual optimizer, as the space of LSC sfunctions ψ does not have "compactness"; unlike the usual OT case where ψ are Lipschitz functions (for Lipschitz costs).

One may still find a reduction to a compact function space to get:
[Ghoussoub/ K./ Palmer]: **Dual attainment:**
Assume: $\mu \prec_{SH} \nu$, $\operatorname{supp}\mu, \operatorname{supp}\nu$ are compact, $\mu \in H^{-1}$,
$0 \le L(t,x) \le D$
($c(x,y) = |x-y|$ or $-M \le \Delta_y c(x,y) \le M$), ...
Then
$\exists$ **optimal dual** $\psi^* \in LSC \cap H_0^1$.

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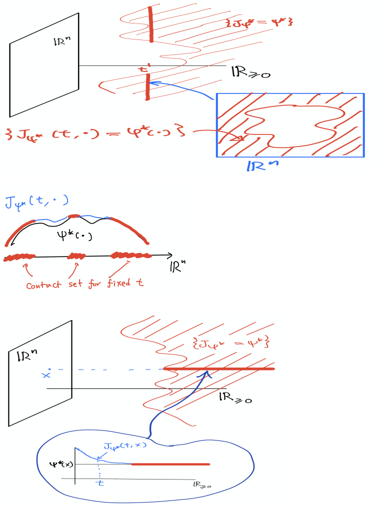
$\exists$ **optimal dual** $\psi^* \in LSC \cap H_0^1$.

Use **the optimal dual** ψ^* to define the **Barrier**:

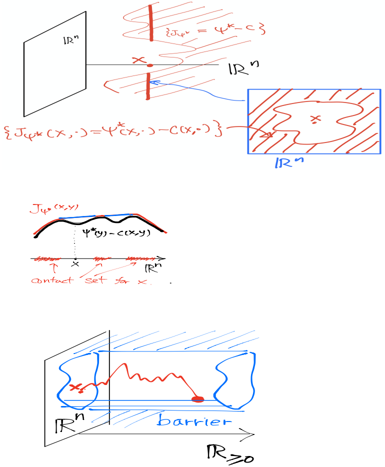
(Markovian) $R^* = \{(x,t) \mid J_{\psi^*}(t,x) = \psi^*(x)\} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$.

(NonMarkovian) $R^* = \{(x,y) \mid J_{\psi^*}(x,y) = \psi^*(y) - c(x,y)\} \subset \mathbb{R}^n \times \mathbb{R}^n$.

(Markovian)



(NonMarkovian)



Optimal stopping $=$ Hitting time to the Barrier

Assume

- (Markovian case) $t \mapsto L(t,x) \nearrow (t \mapsto L(t,x) \searrow)$ strictly.
- (Non-Markovian case) $c(x,y) = |x-y|$ or $d(x,y)$ Riemannian distance, among others.

[Ghoussoub/ K./ Palmer]
Under reasonable assumptions on μ, ν, we have the optimal stopping time τ^* uniquely given by

$$\tau^* = \tau^{R^*}.$$

Corollary: The optimal stopping time is unique.

Eulerian formulation and the barrier (Markovian case)

$$\mathcal{P}(\mu,\nu) = \mathcal{P}_1(\mu,\nu) := \inf_{(\eta,\rho)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^+} L(t,x)\eta(dt,dx)$$

$$\text{subject to } \rho(t,x) + \partial_t \eta(t,x) = \frac{1}{2}\Delta\eta(t,x),$$

$$\int_{\mathbb{R}^+} d\rho = \nu, \qquad \eta(0,x) = \mu(x).$$

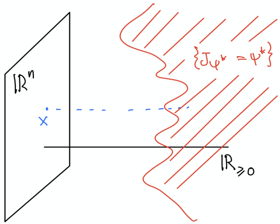
The unique optimal solution (η^*, ρ^*) is a weak solution to the PDE, determined by the condition

$$\eta^*(R^*) = 0 \text{ and } \rho^*(R^*) = 1.$$

Moreover, $\forall g \in C_c(\mathbb{R}^+ \times \mathbb{R}^n)$,

$$\mathbb{E}\big[g(\tau^*, B_{\tau^*})\big] = \int_{\mathbb{R}^n} \int_{\mathbb{R}^+} g(t,x)\rho^*(dt,dx),$$

$$\mathbb{E}\Big[\int_0^{\tau^*} g(t,B_t)dt\Big] = \int_{\mathbb{R}^n} \int_{\mathbb{R}^+} g(t,x)\eta^*(dt,dx).$$



Part II: Optimal Brownian stopping with free target under density constraint.

[I.Kim & K., Work in progress]

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Let $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (e.g. $f \equiv 1$).

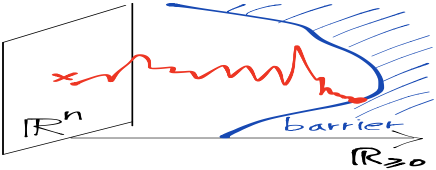
$$\mathcal{P}_f(\mu) := \inf_{\tau}\{\mathcal{C}(\tau) \mid B_0 \sim \mu,\ B_\tau \sim \nu,\ \nu \leq f\}$$

- Also motivated from the **(Non-stochastic case)**

$$\mathcal{P}_f(\mu) := \inf_{\nu}\{W_2^2(\mu,\nu)\ | \nu \leq f\}$$

  of [De Philippis/Mészáros/Santambrogio/Velichkov '15] BV Estimates in Optimal Transportation ...

- We focus on the (Markovian) cost $\mathbb{E}\left[\int_0^\tau L(t, B_t)dt\right]$. Barriers look like



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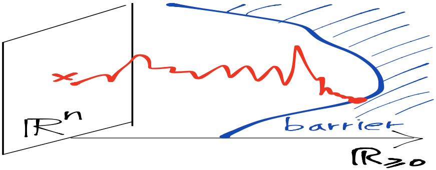

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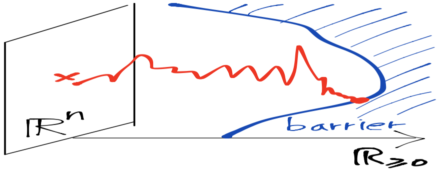

Existence of optimal τ^*

[I.Kim & K., Work in progress]

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Easy existence by assuming $\operatorname{supp} f$ is compact.
We will get optimal τ^*, $B_{\tau^*} \sim \nu^*$ such that τ^* is the unique optimal solution for $P(\mu, \nu^*)$.

- **Can apply results for fixed target problem:**
  dual attainment ψ^*, barrier R^*, hitting time, Eulerian formulation, etc.

Much less clear if $f \equiv 1$.

- How do we know that the mass will not spread to infinity?
  - Use the compact support case then take limit.
  - To control the limit, use tools like **monotonicity/saturation**.

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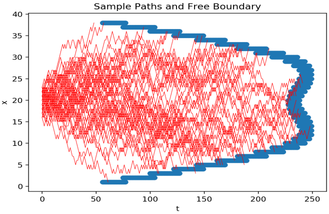
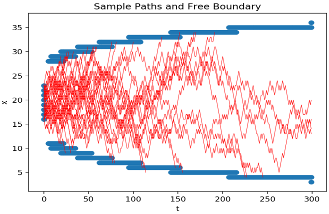
Two cases: D1 and D2

(D1) $t \mapsto L(t,x) \nearrow$ striclty
(D2) $t \mapsto L(t,x) \searrow$ strictly



(D1) hitting from below



(D2) hitting from above

$$s(x) := \begin{cases} \inf\{t \in \mathbb{R}^+;\ J_{\psi^*}(t,x) = \psi^*(x)\} & \text{for (D1)} \\ \sup\{t \in \mathbb{R}^+;\ J_{\psi^*}(t,x) = \psi^*(x)\} & \text{for (D2)} \end{cases}$$

$$\tau^* = \begin{cases} \inf\{t \mid t \geq s(B_t)\} & (D1) \\ \inf\{t \mid t \leq s(B_t)\} & (D2) \end{cases}$$

Monotonicity

[I.Kim & K., Work in progress]

$$\mathcal{P}_f(\mu) := \inf_\tau \{\mathcal{C}(\tau) \mid B_0 \sim \mu,\ B_\tau \sim \nu,\ \nu \le f\}$$

Assume

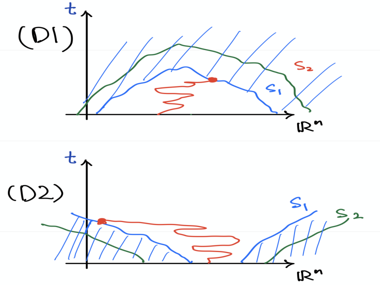
- either (D1) or (D2).
- τ_i be optimal solution for $\mathcal{P}_f(\mu_i)$, hitting the space-time boundary s_i, $i = 1, 2$;
- $B_{\tau_i} \sim \nu_i$, $i = 1, 2$.

If $\mu_1 \le \mu_2$, **then**

$$\tau_1 \le \tau_2 \text{ a.s., and } \nu_1 \le \nu_2;$$

$s_1 \le s_2$ in (D1), $s_1 \ge s_2$ in (D2).

(Monotonicity)



L^1-contraction/uniqueness/BV-estimate

[I.Kim & K., Work in progress]

Corollary (Without assuming $\mu_1 \leq \mu_2$)

Under (D1) or (D2), for the optimal solutions of

$$\mathcal{P}_f(\mu) := \inf_\tau \{\mathcal{C}(\tau) \mid B_0 \sim \mu,\ B_\tau \sim \nu,\ \nu \leq f\}$$

we have

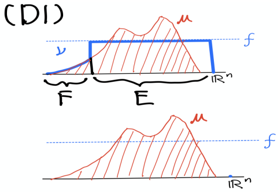
- *(L^1-contraction)* $\|\nu_1 - \nu_2\|_{L^1} \leq \|\mu_1 - \mu_2\|_{L^1}$.
- *(Uniqueness) Optimal ν (thus τ) is uniquely determined.*
- *(BV-estimate) (If $f \equiv$ const)*
  $\|\nu_i\|_{BV} := \int |\nabla \nu_i| \leq \|\mu\|_{BV} := \int |\nabla \mu_i|$.

Saturation.

[I.Kim & K., Work in progress]:
Assume: (D1) or (D2) & the optimal solution $B_{\tau^*} \sim \nu$ to $\mathcal{P}_f(\mu)$.

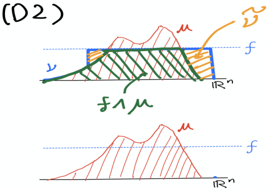
(D1) $\nu = f\chi_E + \mu\chi_F$

- $|E \cap F| = 0$;
- in E the Brownian motion does not stop immediately;
- in F the Brownian paths stop immediately.



(D2) $\nu = \tilde{\nu} + f \wedge \mu$

- $\tilde{\nu}$ optimal for $\mathcal{P}_{\tilde{f}}(\tilde{\mu})$ with $\tilde{\mu} = \mu - f \wedge \mu$, and $\tilde{f} = f - f \wedge \mu$;
- $\tilde{\nu} = \tilde{f}\chi_E$ for some set E.
- for the portion $f \wedge \mu$ the Brownian motion stops immediately.



Saturation.

[I.Kim & K., Work in progress]:
Assume: (D1) or (D2) & the optimal solution $B_{\tau^*} \sim \nu$ to $\mathcal{P}_f(\mu)$.

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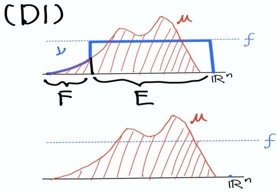

(D2) $\nu = \tilde{\nu} + f \wedge \mu$

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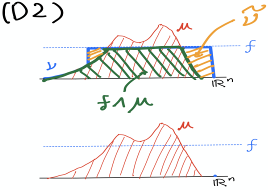
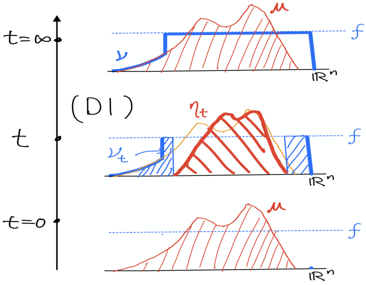


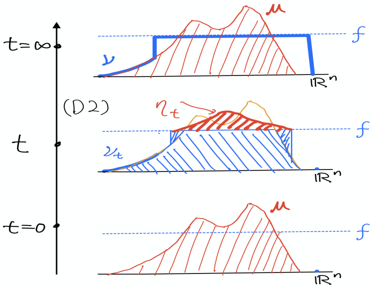
Saturation

[I.Kim & K., work in progress]
Assume: (D1) or (D2) & the optimal solution $B_{\tau^*} \sim \nu$ to $\mathcal{P}_f(\mu)$.
At time t, $\quad B_{\tau^* \wedge t} \sim \mu_t, \quad \mu_t = \eta_t + \nu_t$
η_t= moving mass, $\quad \nu_t$ = stopped mass.
Then:

(D1)



(D2)

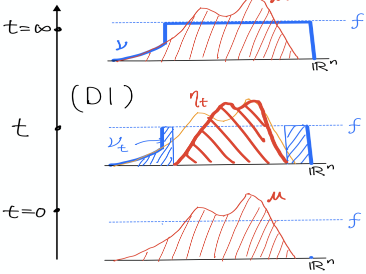


Physical interpretation: the Stefan problem.

η_t= moving mass/particles, ν_t = stopped mass/particles.

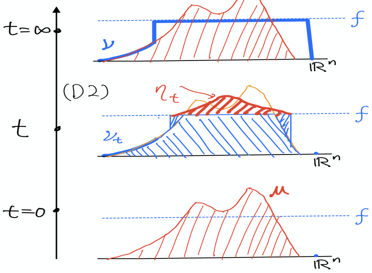
(D1)

Freezing. Ice = $\mathrm{supp}\, \nu_t$.



(D2)

Melting. Water = $\mathrm{supp}\, \eta_t$.



Connection to the Stefan problem

[Ghoussoub/K./Palmer'19] [I.Kim & K., work in progress]

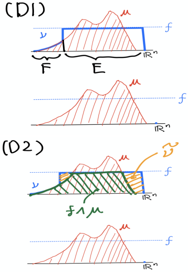
Assume among others, $\mu \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n)$.

(Then, one can show $\nu \in C(\overline{\{\nu > 0\}})$.

Then η is a weak solution of the weighted Stefan problem:

$$\begin{cases} \partial_t \eta - \frac{1}{2}\Delta\eta = 0 & \text{in } \{\eta > 0\}; \\ V = w(x)\nabla\eta \cdot \vec{n} & \text{on } \partial\{\eta > 0\}, \end{cases}$$

$$w(x) = \begin{cases} -\tilde{\nu}^{-1}(x) & \text{(D1) "freezing"}, \\ +\tilde{\nu}^{-1}(x) & \text{(D2) "melting"}. \end{cases}$$



The free boundary $\partial\{\eta > 0\} \subset \mathbb{R}^n$ at each time.

$V =$ the normal velocity of $\partial\{\eta > 0\}$.

$\vec{n} =$ outward unit normal.

- $\tilde{\nu} = f$ for (D1) and $\tilde{\nu} = f - f \wedge \mu$ for (D2).
- The initial data $\eta_0 = \mu - \mu\chi_F$ for (D1), $\eta_0 = \mu - f \wedge \mu$ for (D2).
- F (the inactive region) is determined by μ, ν.
- F is disjoint from the initial active region of the flow, the set $E = \overline{\{\eta > 0\}} \cap \{t = 0\}$.

Connection to the Stefan problem

[Ghoussoub/K./Palmer'19] [I.Kim & K., work in progress]

Assume among others, $\mu \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n)$.
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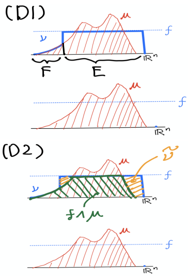

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- F is disjoint from the initial active region of the flow, the set $E = \overline{\{\eta > 0\}} \cap \{t = 0\}$.

The Stefan problem

- Only few on (D1) the freezing problem:
  - 1-D, special case [Chayes/Swindle][Chayes/I.Kim'08]
  - self-similar, well-prepared (smooth) radial data [Hadzic/Raphael]
- Many works for (D2) melting problem
  - Well-posedness: [Meirmanov] .....
  - Regularity: [Caffarelli'77], [Athanasopoulos/Cafferlli/Salsa '96-98][Choi-I.Kim'10], ...
  - Stability: [Hadzic/Shkoller'14][Hadzic/Raphael'16], ...

Challengies: Regularity of the free boundary!

- Breakthrough of [Caffarelli ’77]
- Many other important works...
- The recent breakthrough of [Alessio Figalli] (works with J. Serra, X. Ros-Otton,....);
- Also [De Philippis/ Spolaor/ Velichkov].

BV estimates at $t = \infty$ and at $t < \infty$.

[I.Kim/ K., Work in Progress]

- Let τ^* be optimal for $\mathcal{P}_f(\mu)$ and $f \equiv 1$.
- Let $B_{\tau^* \wedge t} \sim \mu_t$ and $\mu_t = \nu_t + \eta_t$,
  where $\nu_t =$ stopped mass, $\quad \eta_t =$ moving mass.
- Note: from the saturation property, regularity of $\partial\{\eta > 0\}$ is related to regularity of ν_t.
- For the case $t \to \infty$, $\nu = \lim_{t\to\infty} \mu_t$, we now have our BV estimate:

$$\|\nu\|_{BV} \leq \|\mu\|_{BV}.$$

Question: What about for $t < \infty$?

Answer: We have

- (D1): **??**
- (D2): new proof of known facts:

$$\|\nu_t\|_{BV} + \|\eta_t\|_{BV} \leq \|\mu\|_{BV}.$$

Star shapes and Lipschitzness of the free boundary.

[I.Kim & K., Work in progress]

Corollary (of Monotonicity)

Assume (D1) *or (D2) and* $f \equiv 1$.

Let

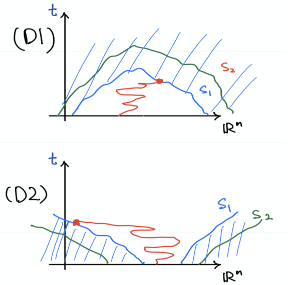
$A = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mid \eta(x,t) > 0\} \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$,

"the active region",

and consider the free boundary $\partial A \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$.

- **If** μ *is radially decreasing with respect to each point* $x_0 \in B_r$ *for a small ball* B_r,*(e.g.* $\mu = \chi_S$ *for a Lipschitz star-shaped set S)* **then** ∂A *is* **Lipschitz**.

(Monotonicity)



Works on [Brownian stopping]

- **[Fixed target]**
  - [Markovian]. $\mathcal{C}(\tau) = \mathbb{E}[\int_0^{\tau} L(t, B_t)dt]$.
    - [Ghoussoub/ K. / Palmer] PDE methods for Skorokhod embeddings. *Calc. Var. PDE (2019)*
  - [Non-Markovian]. $\mathcal{C}(\tau) = \mathbb{E}[|B_0 - B_\tau|]$.
    - [Ghoussoub / K. / Lim] Optimal Brownian Stopping between radially symmetric marginals in general dimensions. To appear in *SIAM J. Control & Optimization*.
    - [Ghoussoub/ K. / Palmer] A solution to the Monge transport problem for Brownian martingales To appear in *Ann. of Probability*.
    - [Ghoussoub/ K. / Palmer] Optimal stopping of stochastic transport minimizing submartingale costs. *Arxiv e-prints,* 2020.
- **[Free target under density constraints]**
  - [Inwon Kim/ K.] Work in progress.

Works on [Other cases with fixed target]

- [Martingale Transport]
  - [Ghoussoub / K. / Lim] Structure of optimal martingale transport plans in general dimensions. *Ann. of Probability.* 2019.
- [Stopping with control/drifts] $\mathcal{C}(\tau) = \mathbb{E}[\int_0^{\tau} L(t, X_t, A)dt]$.
  - [Deterministic] $dX_t = Adt$.
    - [Ghoussoub/ K. / Palmer] Optimal transport with controlled dynamics and free end times. *SIAM Journal on Control and Optimization*, 2018.
  - [Stochastic] $dX_t = Adt + dB_t$.
    - [Dweik/ Ghoussoub/ K. / Palmer] Stochastic optimal transport with free end time. To appear in *Ann. Institut Henri Poincaré (B)*

Thank you very much!