The Bernstein problem for equations of minimal surface type

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Partly joint work with Y. Yang
The Bernstein Problem

**Theorem (Bernstein, 1915)**

Assume \( u \in C^2(\mathbb{R}^2) \) solves the minimal surface equation

\[
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.
\]

Then \( u \) is linear.

- Different from linear case (many entire harmonic functions)

**Bernstein Problem:**

Prove the same result in higher dimensions, or construct a counterexample.
Solution to the Bernstein problem:

- $n = 2$ (Bernstein, 1915): Topological argument

- New proof (Fleming, 1962): Monotonicity formula, nontrivial solution in $\mathbb{R}^n \Rightarrow$ non-flat area-minimizing hypercone $K \subset \mathbb{R}^{n+1}$

- $n = 3$ (De Giorgi, 1965): $K = C \times \mathbb{R}$

- $n = 4$ (Almgren, 1966), $n \leq 7$ (Simons, 1968): Stable minimal cones are flat in low dimensions

- $n \geq 8$ (Bombieri-De Giorgi-Giusti, 1969): Counterexample!
The Bernstein Problem

- \( \det D^2w < 0 \) in \( \mathbb{R}^2 \) \( \Rightarrow \) tangent planes to graph \( (w) \) disconnect it into \( \geq 4 \) unbounded pieces

- \( \text{Cor:} \) \( \frac{a_{ij}(x)}{\text{pos}} w_{ij} = 0 \) in \( \mathbb{R}^2 \) \( \Rightarrow \) \( w = \text{const.} \)

- Apply to \( w = \frac{\tan^{-1}(u)}{\text{harmonic on graph } (u)} \)
The Bernstein Problem

\[ K \subset \mathbb{R}^3 \text{ cone + minimal } \Rightarrow \text{ flat} \]

(only 1 nonzero curvature)
\[ K = C \times \mathbb{R} \]

non-flat area-min.

cone in \( \mathbb{R}^n \)
Bernstein’s theorem generalizes to all dimensions with growth hypotheses:

- $|\nabla u| < C$ (De Giorgi, Nash; 1958)
- $u(x) < C(1 + |x|)$ (Bombieri-De Giorgi-Miranda, 1969)
- $|\nabla u(x)| = o(|x|)$ (Ecker-Huisken, 1990)

Some beautiful open problems:

- Do all entire solutions of the MSE have polynomial growth?
- Does there exist a nonlinear polynomial that solves the MSE?
Object of interest: $\Sigma \subset \mathbb{R}^{n+1}$ oriented hypersurface, minimizes

$$A_\Phi(\Sigma) := \int_\Sigma \Phi(\nu) \, dA.$$ 

Here $\nu$ = unit normal, and $\Phi$ is 1-homogeneous, positive and $C^{2,\alpha}$ on $\mathbb{S}^n$, and $\{\Phi < 1\}$ uniformly convex (“uniform ellipticity”)

**E-L Equation:** $\Phi_{ij}(\nu) II_{ij} = 0$ (“balancing of principal curvatures”)

**$\Phi$-Bernstein Problem:**
If $\Sigma$ is the graph of a function $u : \mathbb{R}^n \to \mathbb{R}$, is it necessarily a hyperplane?
Φ-Bernstein Problem

\[
\Phi_{ij}(\nu) \Pi_{ij} = 0 \quad \iff \quad \epsilon_{ij}(\nabla \nu) \partial_{ij} = 0, \quad \text{with } \epsilon(p) := \Phi(-p, 1)
\]

\[
\text{Ellipticity degenerates as } \|\nabla \nu\| \to \infty
\]

Derivation:
\[
\int \Phi(\nu) \, dA = \int \Phi \left( \frac{-\nabla \nu}{1 + \|\nabla \nu\|^2} \right) \sqrt{1 + \|\nabla \nu\|^2} \, dx = \int \epsilon(\nabla \nu) \, dx
\]
Positive results:

- $n = 2$ (Jenkins, 1961): $\nu$ is quasiconformal
- $n = 3$ (Simon, 1977): Regularity theorem of Almgren-Schoen-Simon (1977) for parametric problem
- $n \leq 7$ if $\|\Phi - 1\|_{C^{2,1}(\mathbb{S}^n)}$ small (Simon, 1977)
- $|\nabla u| < C$ (De Giorgi-Nash) or $|u(x)| < C(1 + |x|)$ (Simon, 1971)

**Question:** $4 \leq n \leq 7$ ??
Theorem (M., 2020)

There exists a quadratic polynomial on $\mathbb{R}^6$ whose graph minimizes $A_\Phi$ for a uniformly elliptic integrand $\Phi$.

- $\Phi$ necessarily far from 1 on $S^6$ (level sets “box-shaped”)
- The analogous quadratic polynomial does not work in $\mathbb{R}^4$
- Open: $n = 4, 5$
Approach of Bombieri-De Giorgi-Giusti ($\Phi|_{S^{n-1}} = 1$):

Let $(x, y) \in \mathbb{R}^8$ with $x, y \in \mathbb{R}^4$, and let $C := \{|x| = |y|\}$

- Find a smooth perturbation $\Sigma$ of the Simons cone $C$, whose dilations foliate one side (ODE analysis)

- Notice that $\Sigma \sim \{r^3 \cos(2\theta) = 1\}$ far from the origin (here $r^2 = |x|^2 + |y|^2$, $\tan \theta = |y|/|x|$)

- Build global super/sub-solutions $\sim r^3 \cos(2\theta)$ in $\{|x| > |y|\}$ (hard), solve Dirichlet problem in larger and larger balls
$C = \{ |x| = |y|^{3/2} \}$

$\Sigma'$ - minimal

$R$, $R^2$, $R^4$, $x$, $y$
The case $K \leq 3$
Our approach: Fix $u$, build $\Phi$

- Equation is $\varphi_{ij}(\nabla u)u_{ij} = 0$ (here $\varphi(p) := \Phi(-p, 1)$), rewrite in terms of Legendre transform $u^*$ of $u$ as

$$ (u^*)^{ij} \varphi_{ij} = 0 $$

(a linear hyperbolic eqn for $\Phi$)

- Let $(x, y) \in \mathbb{R}^{2k}$, $x, y \in \mathbb{R}^k$, $u = \frac{1}{2}(|x|^2 - |y|^2)$, $\varphi = \psi(|x|, |y|)$

Equation becomes

$$ \Box \psi + (k - 1) \nabla \psi \cdot \left( \frac{1}{s}, -\frac{1}{t} \right) = 0 $$

in positive quadrant (here $|x| = s$, $|y| = t$, $\Box = \partial^2_s - \partial^2_t$)
The case $k = 3$ is special:

- Equation reduces to $\Box (st \psi) = 0$, so

  $$
  \psi(s, t) = \frac{f(s + t) + g(s - t)}{st}
  $$

- Choose $f, g$ carefully s.t. $\Phi$ is uniformly elliptic (tricky part)

One choice of $\Phi$ is

$$
\Phi(p, q, z) = \frac{((|p| + |q|)^2 + 2z^2)^{3/2} - ((|p| - |q|)^2 + 2z^2)^{3/2}}{2^{5/2}|p||q|},
$$

with $p, q \in \mathbb{R}^3$ and $z \in \mathbb{R}$. 
Φ-Bernstein Problem

\[ \{ \Phi = \| \mathbf{x} \|^2 \wedge \{ x_3 = 0 \} \} \]
Some remarks:

- There are many possible choices of $\Phi$ (perturb $f$, $g$)

- $\{u = \text{const.}\}$ minimize $A_{\Phi_0}$, $\Phi_0 = \Phi|_{\{x_7=0\}}$ (homogeneity of $u$)

- The case $u = \frac{1}{2}(|x|^2 - |y|^2)$, $k = 2$: By above remark, $\{u = 1\}$ must minimize a uniformly elliptic functional. This is false when $k = 2$ (symmetries of $u + \text{ODE analysis}$)

However, the cone $C := \{u = 0\} \subset \mathbb{R}^4$ minimizes a uniformly elliptic functional (Morgan 1990, proof by calibration technique)...
Current Work (joint with Y. Yang)

An approach in the case $n = 4$: combine the previous ones

1. Proof by “foliation” of Morgan’s result:

**Theorem (M.-Yang, 2020)**

There exist analytic elliptic integrands $\Phi$ on $\mathbb{R}^4$ such that each side of $C$ is foliated by $A_\Phi$-minimizing hypersurfaces.

Furthermore, these hypersurfaces resemble level sets of $\gamma$-homogeneous functions, for any $\gamma \in (1, 3/2)$.

2. Fix an entire function $u$ on $\mathbb{R}^4$ that is asymptotically $\gamma$-homogeneous with $\gamma \in (1, 3/2)$, prove that its graph minimizes a uniformly elliptic functional ($\gamma = 4/3$ looks particularly inviting)
Current Work (joint with Y. Yang)

Controlled growth question:

- Positive result if $|\nabla u|$ grows slowly enough (e.g. $|\nabla u| = O(|x|^\epsilon)$ with $\epsilon(n, \Phi)$ small)?

Regularity of $\Phi$:

- In the 6D example, $\Phi \in C^{2,1}(S^6)$. Can we make $\Phi \in C^\infty(S^n)$? Analytic on $S^n$?
Thank you!