The stickiness property of nonlocal minimal surfaces

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Outline

Nonlocal minimal surfaces

Energy functional dealing with *"pointwise interactions"* between a given set and its complement

Main idea: the "surface tension" is the byproduct of long-range interactions

Implications: nonlocal phase transitions and nonlocal capillarity theories

New effects due to the long-range interactions

Contributions from "far-away" can have a significant influence on the local structures of these new objects

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The fractional perimeter functional

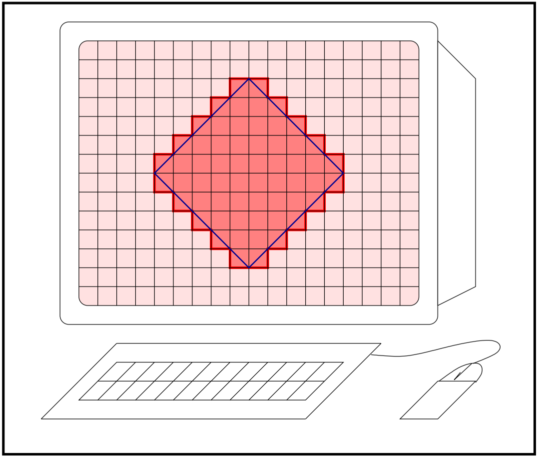
Given $s \in (0, 1)$ and a bounded open set $\Omega \subset \mathbb{R}^n$ with $C^{1,\gamma}$-boundary, the s-perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ in Ω is defined as

$$\begin{aligned} \mathrm{Per}_s(E;\Omega) &:= L(E\cap\Omega, (\mathcal{C}E)\cap\Omega) \\ &\quad + L(E\cap\Omega, (\mathcal{C}E)\cap(\mathcal{C}\Omega)) + L(E\cap(\mathcal{C}\Omega), (\mathcal{C}E)\cap\Omega), \end{aligned}$$

where $\mathcal{C}E = \mathbb{R}^n \setminus E$ denotes the complement of E, and $L(A, B)$ denotes the following nonlocal interaction term

$$L(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\,dx\,dy \qquad \forall A, B \subseteq \mathbb{R}^n,$$

This notion of s-perimeter and the corresponding minimization problem were introduced in [Caffarelli-Roquejoffre-Savin, 2010].

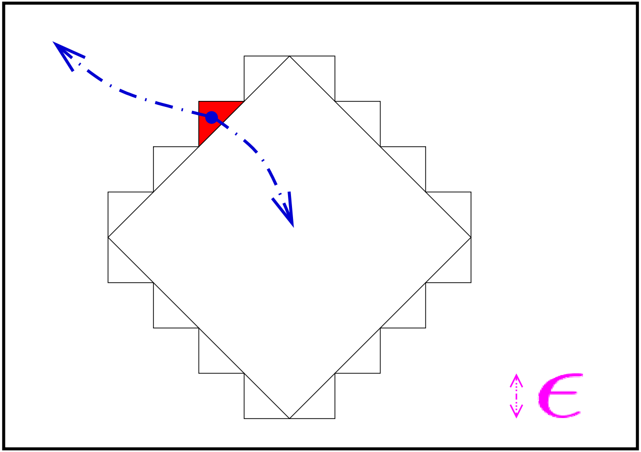


Side 1.

Perimeter 4.

Approximate Perimeter $4\sqrt{2}$.

Error $4(\sqrt{2}-1)$.



Error in each pixel $O(\epsilon^{2-s})$.
Number of pixels $O(\epsilon^{-1})$
Error $O(\epsilon^{1-s})$.

[Caffarelli-Roquejoffre-Savin, 2010]

1) Existence theorem:
   there exists E s-minimizer for Per_s in Ω with $E \setminus \Omega = E_0 \setminus \Omega$.

2) Maximum principle:
   E s-minimizer and $(\partial E) \setminus \Omega \subset \{|x_n| \leqslant a\} \Rightarrow \partial E \subset \{|x_n| \leqslant a\}$.

3) If ∂E is an hyperplane, then E is s-minimizer.

4) If E is s-minimizer in B_1, then ∂E is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set Σ, with Hausdorff dimension less or equal than $n-2$.

5) If E is s-minimizer and $0 \in \partial E$, then

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y|^{n+s}}\, dy = 0.$$

[Caffarelli-Roquejoffre-Savin, 2010]

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Regularity in dimension 2

[Savin-Valdinoci, 2013]:

Regularity of cones in dimension 2.

If E is s-minimizer in B_1, then ∂E is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set Σ, with Hausdorff dimension less or equal than $n-3$.

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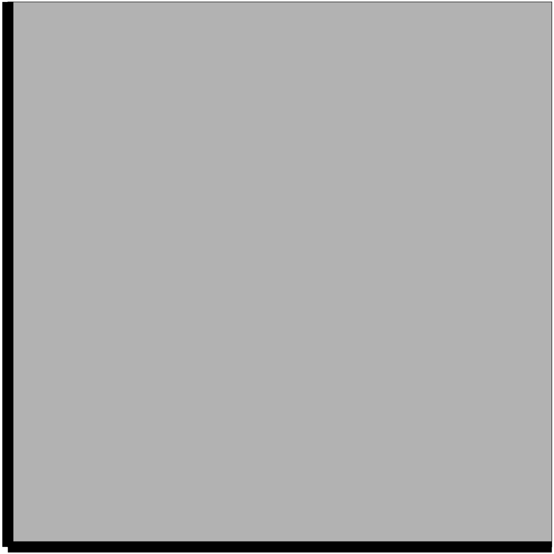

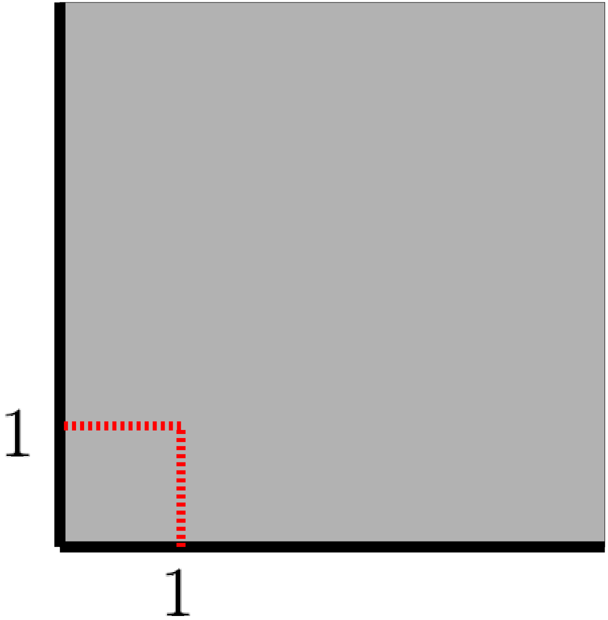
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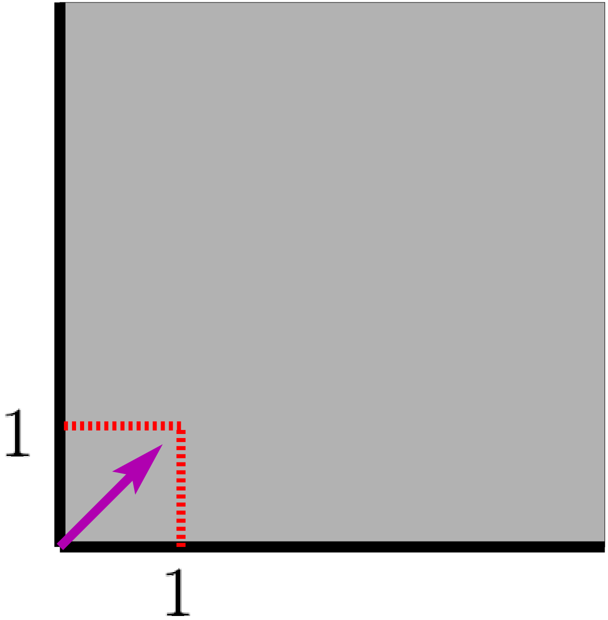
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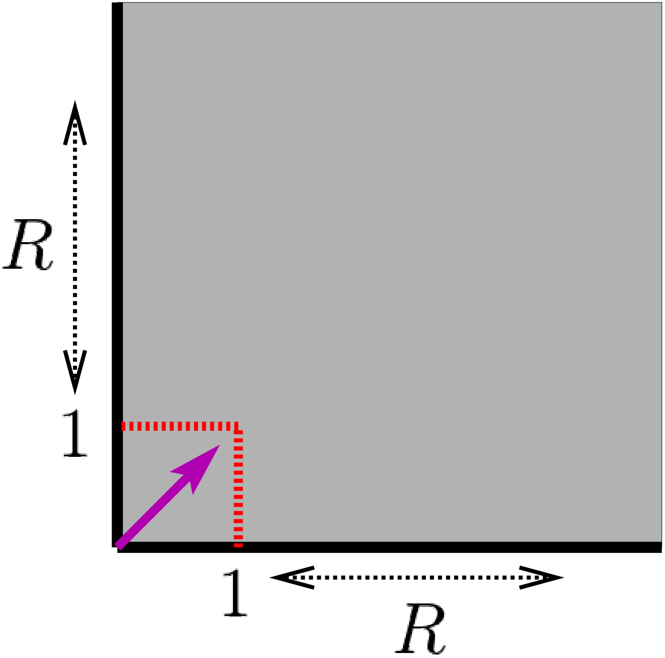
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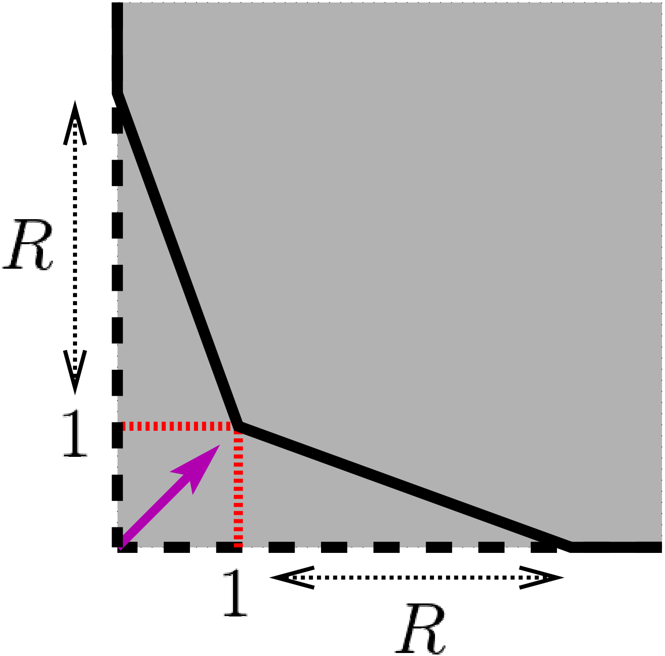
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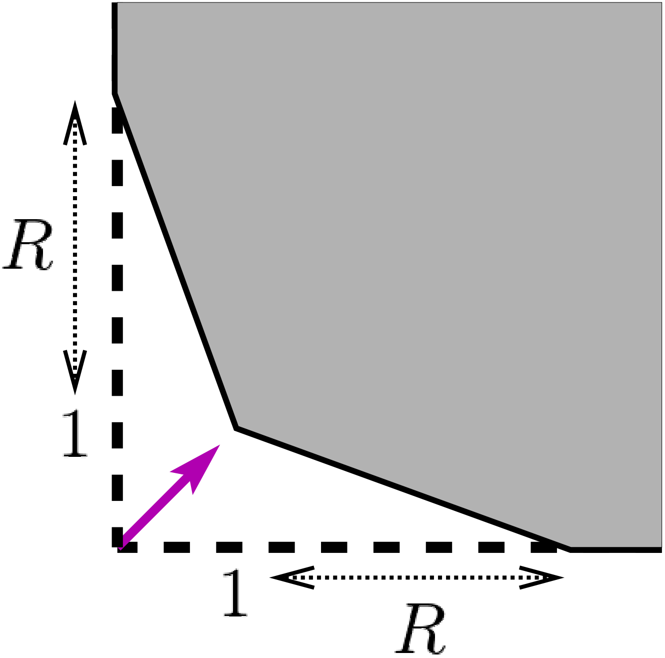
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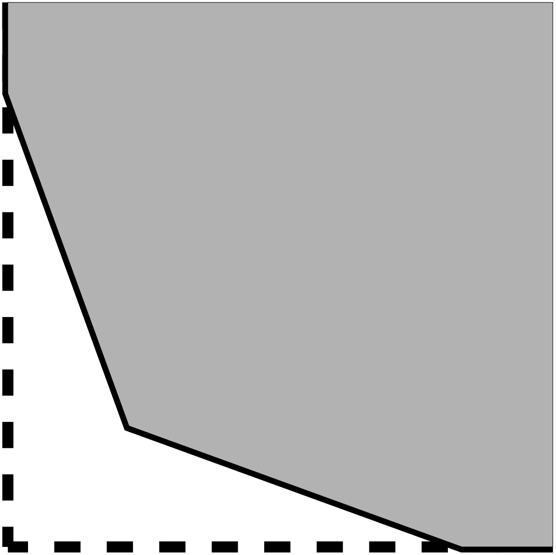

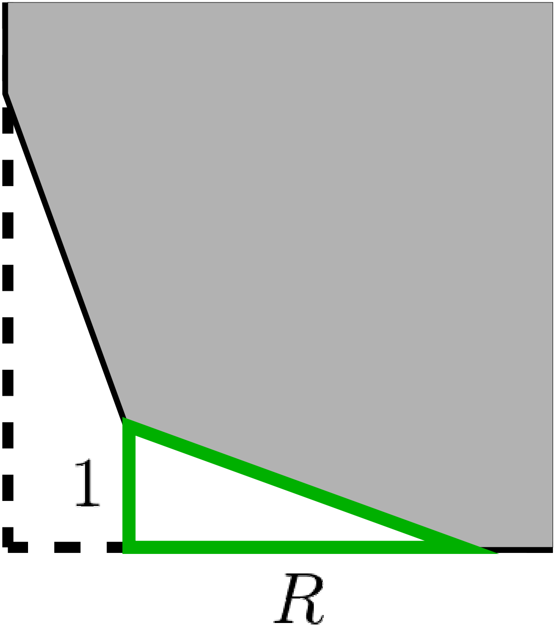
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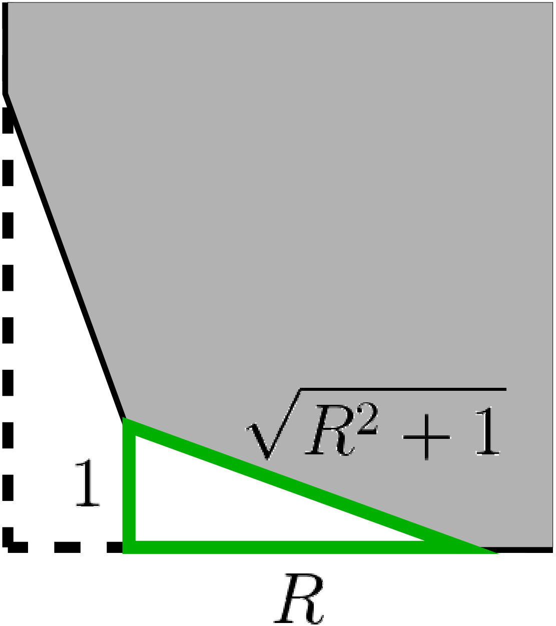
$\sqrt{R^2+1}$
1
R

Regularity for graphs in dimension 3

[Figalli-Valdinoci, 2013]:

Bernstein-type result:

- E is s-minimal in $\mathbb{R}^{n+1}$ and ∂E is a global graph,
- s-minimal surfaces are smooth in $\mathbb{R}^n$

$\Rightarrow \partial E$ is hyperplane.

Regularity of minimal graph in dimension 3.

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Regularity of minimal graph in dimension 3.

Limit as $s \to 1$

[Bourgain-Brezis-Mironescu, 2001], [Dávila, 2002], [Ponce, 2004], [Caffarelli-Valdinoci, 2011], [Ambrosio-De Philippis-Martinazzi, 2011], [Lombardini, 2018]:

$$(1-s)\mathrm{Per}_s \to \mathrm{Per}, \qquad \text{as } s \nearrow 1$$

(up to normalizing multiplicative constants).

$$\Downarrow$$

[Caffarelli-Valdinoci, 2013]:
s close to 1: nonlocal minimal surfaces are as regular as classical minimal surfaces.

(If E is s-minimizer in B_1, then ∂E is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set Σ, with Hausdorff dimension less or equal than $n-8$.)

Limit as $s \to 1$

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Limit as $s \to 0$

[Maz'ya-Shaposhnikova, 2002] and
[Dipierro-Figalli-Palatucci-Valdinoci, 2013]:
If there exists the limit

$$\alpha(E) := \lim_{s \searrow 0} s \int_{E \cap (\mathcal{C}B_1)} \frac{1}{|y|^{n+s}}\, dy,$$

then

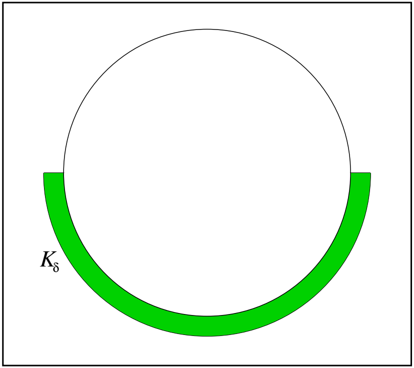
$$\lim_{s \searrow 0} s\,\mathrm{Per}_s(E,\Omega) = \big(\omega_{n-1} - \alpha(E)\big)\,\frac{|E \cap \Omega|}{\omega_{n-1}} + \alpha(E)\,\frac{|\Omega \setminus E|}{\omega_{n-1}}.$$

Stickiness to half-balls

For any $\delta > 0$,

$$K_\delta := \left(B_{1+\delta} \setminus B_1\right) \cap \{x_n < 0\}.$$

We define E_δ to be the set minimizing the s-perimeter among all the sets E such that $E \setminus B_1 = K_\delta$.



Stickiness to half-balls

There exists $\delta_o > 0$ such that for any $\delta \in (0, \delta_o]$ we have that

$$E_\delta = K_\delta.$$

Stickiness to the sides of a box

Given a large $M > 1$ we consider the s-minimal set E_M in $(-1, 1) \times \mathbb{R}$ with datum outside $(-1, 1) \times \mathbb{R}$ given by the jump $J_M := J_M^- \cup J_M^+$, where

$$J_M^- := (-\infty, -1] \times (-\infty, -M)$$
$$\text{and} \qquad J_M^+ := [1, +\infty) \times (-\infty, M).$$

Stickiness to the sides of a box

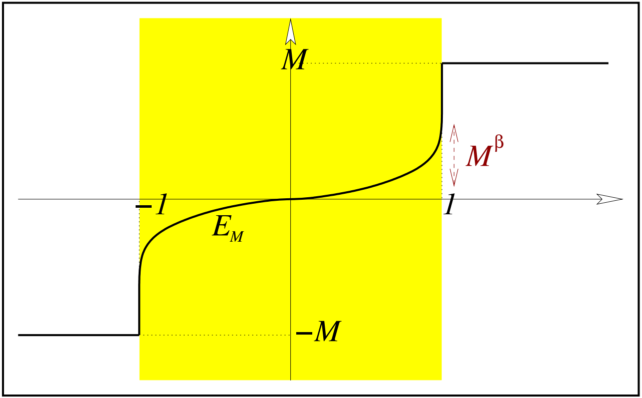
There exist $M_o > 0$ and $C_o \geq C_o' > 0$, depending on s, such that if $M \geq M_o$ then

$$[-1,1) \times [C_o M^{\frac{1+s}{2+s}},\, M] \subseteq E_M^c$$

$$\text{and} \qquad (-1,1] \times [-M,\, -C_o M^{\frac{1+s}{2+s}}] \subseteq E_M.$$

Also, the exponent $\beta := \frac{1+s}{2+s}$ above is optimal.

Stickiness to the sides of a box



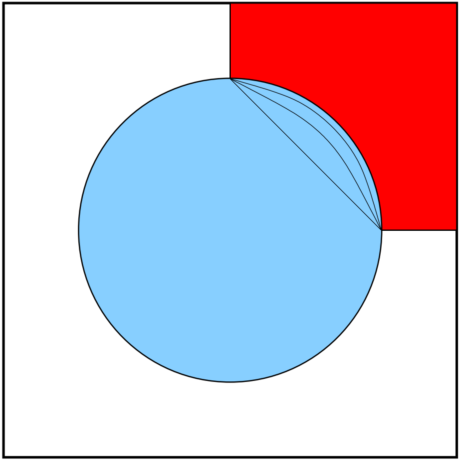
Stickiness as $s \to 0^+$

We consider a sector in $\mathbb{R}^2$ outside B_1, i.e.

$$\Sigma := \{(x, y) \in \mathbb{R}^2 \setminus B_1 \text{ s.t. } x > 0 \text{ and } y > 0\}.$$

Let E_s be the s-minimizer of the s-perimeter among all the sets E such that $E \setminus B_1 = \Sigma$.
Then, there exists $s_o > 0$ such that for any $s \in (0, s_o]$ we have that $E_s = \Sigma$.

Stickiness as $s \to 0^+$



Instability of the flat fractional minimal surfaces

Fix $\epsilon_0 > 0$ arbitrarily small. Then, there exists $\delta_0 > 0$, possibly depending on ϵ_0, such that for any $\delta \in (0, \delta_0]$ the following statement holds true.

Assume that $F \supset H \cup F_- \cup F_+$, where

$$H := \mathbb{R} \times (-\infty, 0),$$

$$F_- := (-3, -2) \times [0, \delta)$$

and

$$F_+ := (2, 3) \times [0, \delta).$$

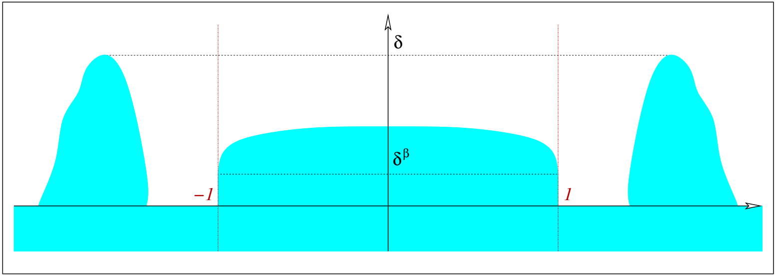
Let E be the s-minimal set in $(-1, 1) \times \mathbb{R}$ among all the sets that coincide with F outside $(-1, 1) \times \mathbb{R}$.

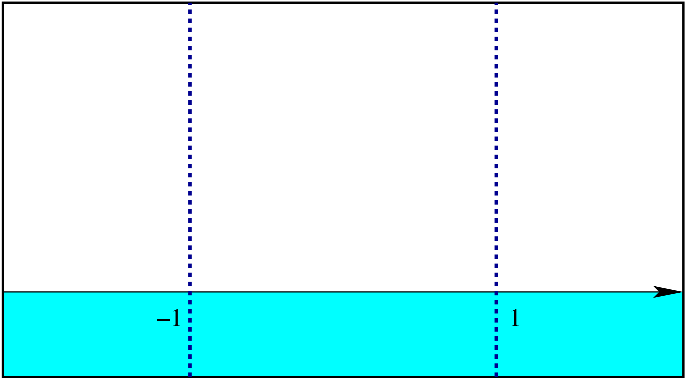
Then

$$E \supseteq (-1, 1) \times (-\infty, \delta^{\frac{2+\epsilon_0}{1-s}}].$$

Instability of the flat fractional minimal surfaces

$\beta := \frac{2+\epsilon_0}{1-s}$

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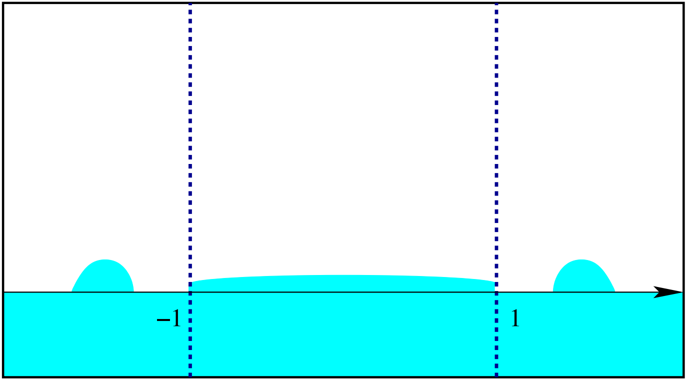
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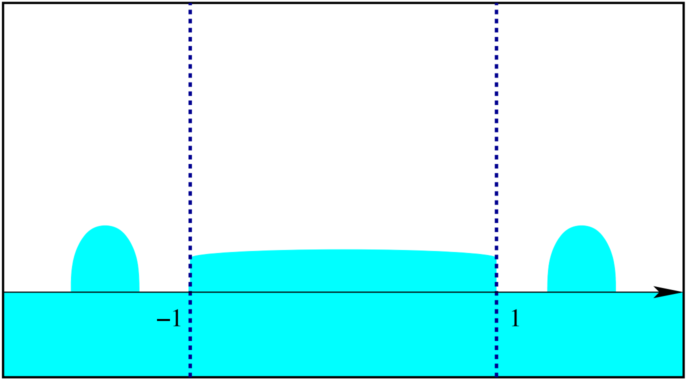
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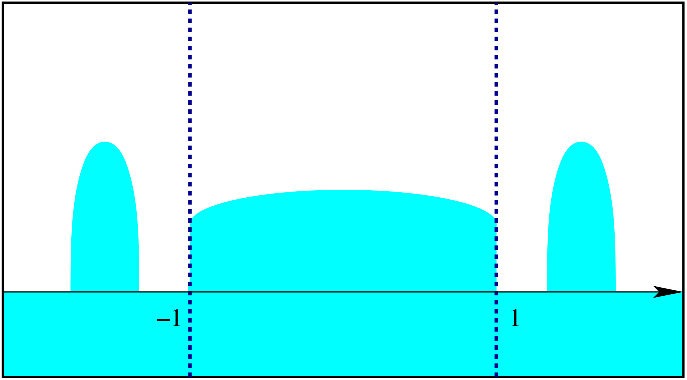
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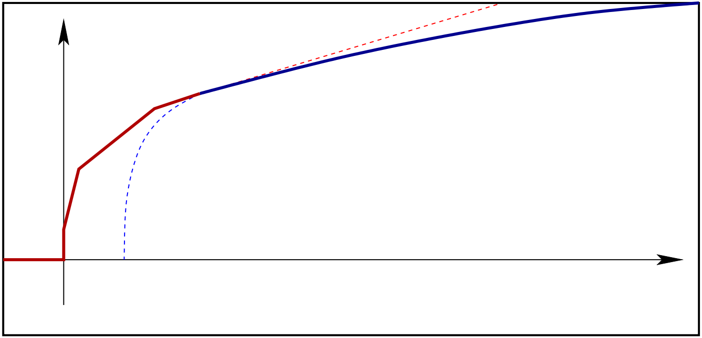
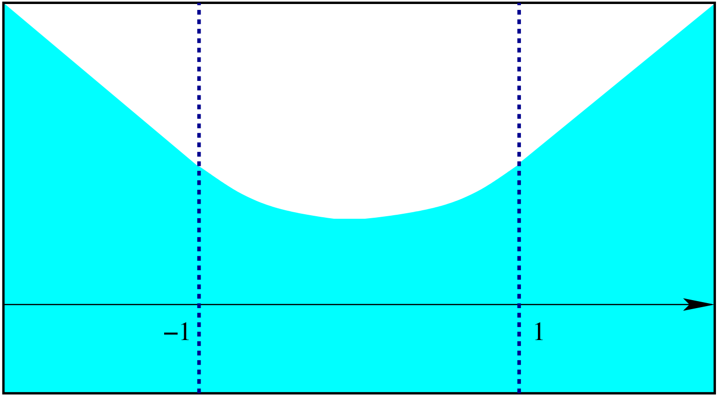
A useful barrier



The usual suspects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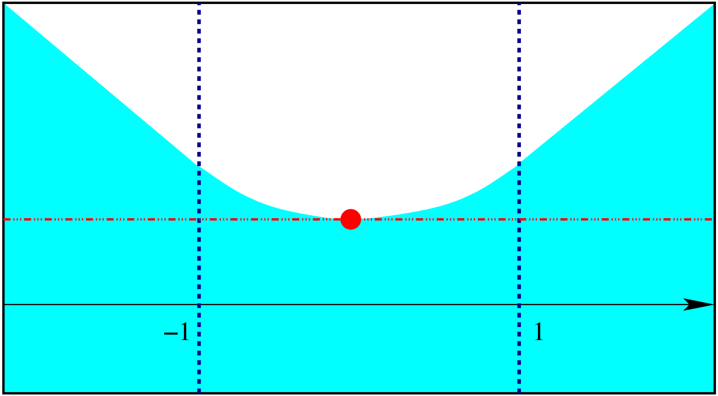

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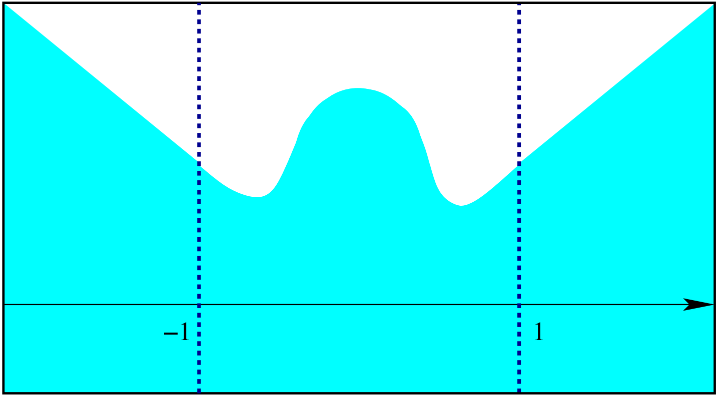
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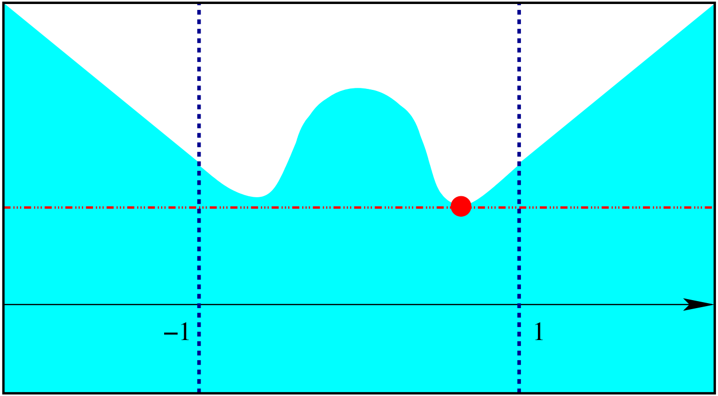

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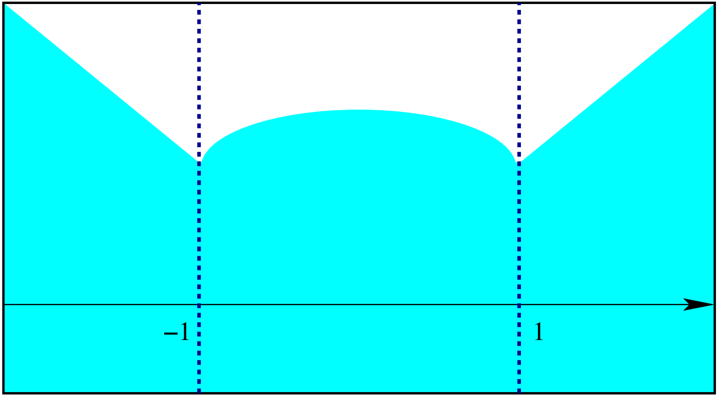
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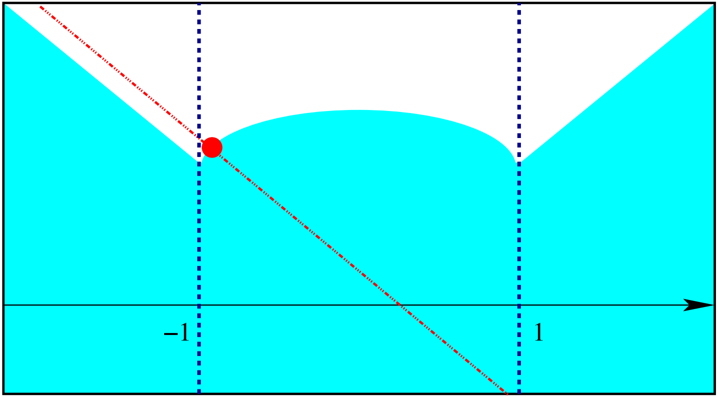
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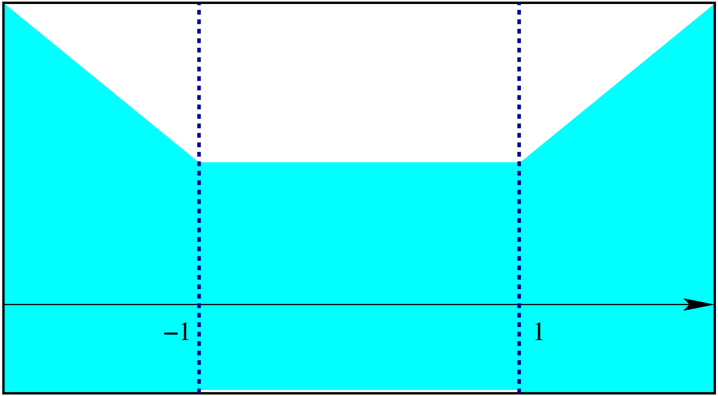

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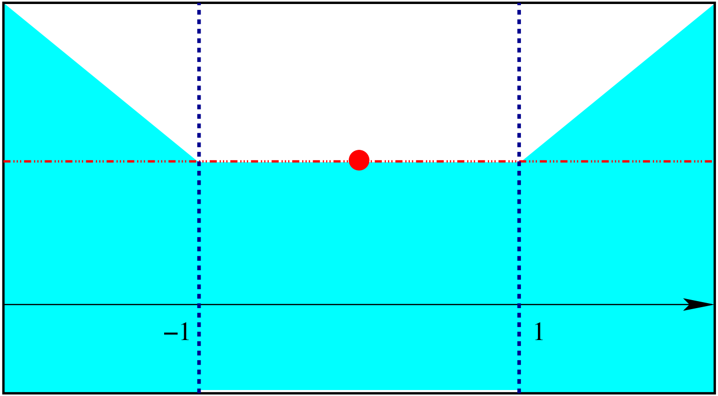
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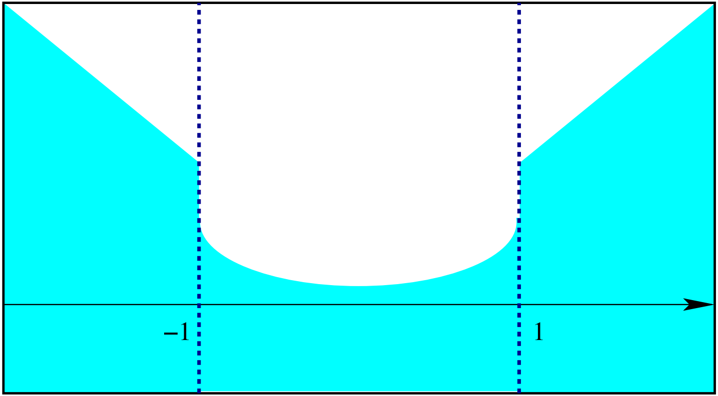
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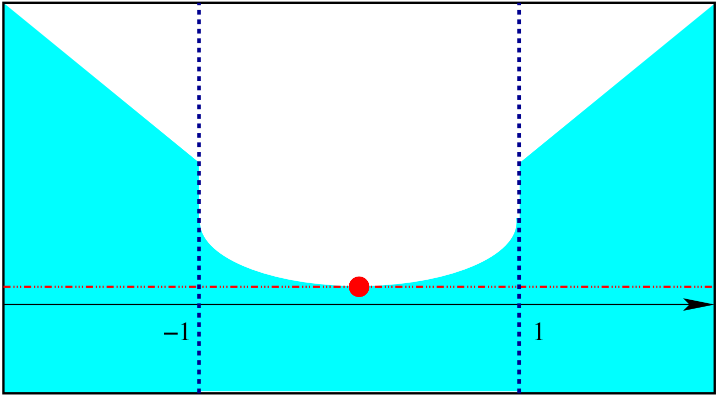
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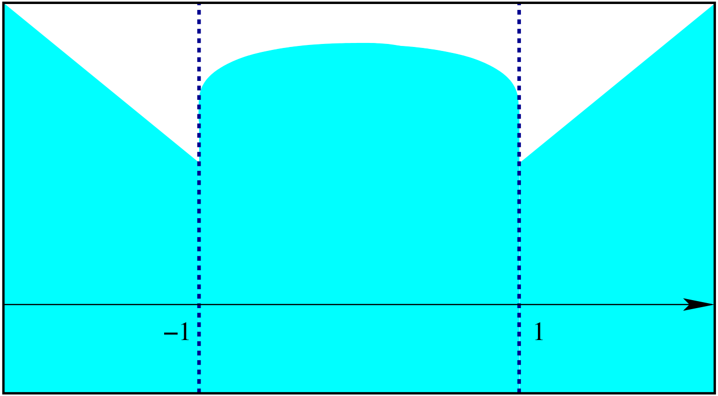

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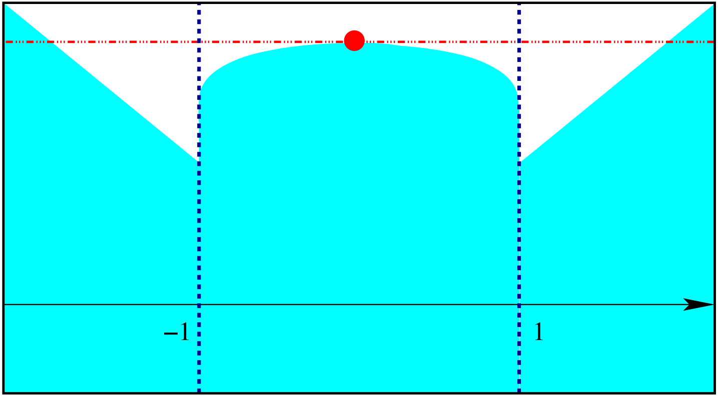

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The leading role of infinity [Bucur-Lombardini-Valdinoci, 2019]

Let $\Omega \subset \mathbb{R}^n$ be bounded, connected and smooth.

Let $E_0 \subset \mathbb{R}^n$ be such that

$$B_r(x_0) \setminus \Omega \subset E_0$$

for some $x_0 \in \partial\Omega$ and $r > 0$, and

$$\alpha(E_0) < \alpha(\text{halfplane}).$$

Then, there exists $s_0 \in (0, 1)$ such that if $s \in (0, s_0)$ and E is the nonlocal minimal set in Ω with external datum E_0, we have

$$E \cap \Omega = \varnothing.$$

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Let $\Omega \subset \mathbb{R}^n$ be bounded, connected and smooth.
Let $E_0 \subset \mathbb{R}^n$ be such that

$$B_r(x_0) \setminus \Omega \subset E_0$$

for some $x_0 \in \partial\Omega$ and $r > 0$, and

$$\alpha(E_0) < \alpha(\text{halfplane}).$$

Then, there exists $s_0 \in (0,1)$ such that if $s \in (0, s_0)$ and E is the nonlocal minimal set in Ω with external datum E_0, we have

$$E \cap \Omega = \varnothing.$$

Three further questions [Dipierro-Savin-Valdinoci, 2020]

1. How regular are the nonlocal minimal surfaces *coming from inside the domain*?

2. Is the Euler-Lagrange equation satisfied *up to the boundary*?

3. How *typical* is the stickiness phenomenon?

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Regularity coming from inside

"Continuity implies differentiability"

Consider a nonlocal minimal graph in $(0, 1)$, with a smooth external graph u_0.

There is a dichotomy:

- either

$$\lim_{x \nearrow 0} u_0(x) \neq \lim_{x \searrow 0} u(x)$$

and

$$\lim_{x \searrow 0} |u'(x)| = +\infty,$$

- or

$$\lim_{x \nearrow 0} u_0(x) = \lim_{x \searrow 0} u(x)$$

and u is $C^{1,\frac{1+s}{2}}$ at 0.

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Some remarks

This dichotomy is a purely nonlinear effect, since the boundary behavior of linear equation is of Hölder type [Serra-Ros Oton].

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Stickiness + dichotomy = butterfly effect

An arbitrarily small perturbation of the flat data produce a boundary discontinuity, which entails an infinite derivative at the boundary.

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As a curve, the nonlocal minimal graph turns out to be always $C^{1,\frac{1+s}{2}}$:

it is either the graph of a $C^{1,\frac{1+s}{2}}$-function (when it is continuous at the boundary!), or it is discontinuous and sticks vertically detaching in a $C^{1,\frac{1+s}{2}}$ fashion [De Silva-Savin] (then the inverse function is a $C^{1,\frac{1+s}{2}}$ function).

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The nonlocal mean curvature can be written in the form

$$\int_{-\infty}^{+\infty} F\left(\frac{u(x+y)-u(x)}{|y|}\right)\frac{dy}{|y|^{1+s}}.$$

And this is a "$C^{1,s}$ operator".

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Boundary Euler-Lagrange equations

If u is a nonlocal minimal graph in $(0, 1)$ with smooth datum outside, then

$$\int_{-\infty}^{+\infty} F\left(\frac{u(x+y)-u(x)}{|y|}\right)\frac{dy}{|y|^{1+s}} = 0$$

for all $x \in [0, 1]$.

With this, we can take any configuration, add an arbitrarily small bump and use the unperturbed configuration as a barrier.

At touching points the additional bump produces an extra-mass violating the Euler-Lagrange equation.

Notice that now also touching at the boundary can be taken into account!

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Stickiness is generic

Let $\varphi \in C_0^\infty([-2,-1],\ [0,1])$, with $\varphi \not\equiv 0$.

Let $u^{(t)}$ be the nonlocal minimal graph in $(0,1)$ with external datum

$$u_0^{(t)} := u_0 + t\varphi.$$

Suppose that

$$\lim_{x \nearrow 0} u_0(x) = \lim_{x \searrow 0} u(x).$$

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Proof of dichotomy

Think about the usual suspects (discontinuous, Lispchitz, Hölder, smooth).

Blow-up.

The “worst” cases to understand are the Hölder and the smooth (the Lispchitz produces non-minimal corners).

The smooth case produces flat objects: use a boundary improvement of flatness (combined with a boundary monotonicity formula) to deduce smoothness of the initial minimizer (for this, use new barrier to go beyond the linear theory!).

The Hölder case produces vertical angles: rule them out by proving that close-to-vertical nonlocal minimal graphs are indeed vertical (for this, slide balls).

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Thank you very much for your attention!