# Stable solutions to semilinear elliptic equations are smooth up to dimension 9

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• Semilinear elliptic PDEs: $-\Delta u = f(u)$ in $\Omega \subset \mathbb{R}^n$, bdd domain

1st variation

Energy: $E_\Omega(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)$ , $F' = f$

2nd variation is $-\Delta - f'(u)$ = linearized operator at $u$ for the equation $-\Delta u = f(u)$

it is nonnegative iff $-\Delta - f'(u) \geq 0$

iff $\int_\Omega f'(u)\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C_c^\infty(\Omega)$ ← Def. of stability

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$\rightarrow$ Competitors $u+\varepsilon\xi$ have all same boundary values as $u$

$\rightarrow$ Our interest: nonlinearities $f$ superlinear at $+\infty$ & $f\geq 0$

$\Downarrow$

NO absolute minimizer exists

$$E_\Omega(t\varphi) = t^2\int_\Omega \frac{1}{2}|\nabla\varphi|^2 - \int_\Omega F(t\varphi) \xrightarrow[t\to+\infty]{} -\infty \quad (F(t\varphi) \gg t^2\varphi^2)$$

- The Barenblatt-Gelfand problem 1963 :

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f(0) > 0$, nondecreasing, convex, & superlinear at $+\infty$.

Model nonlinearities: $f(u) = e^u$ (combustion theory)

$f(u) = (1+u)^p$, $p > 1$

- **The Barenblatt-Gelfand problem 1963:**

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with $f(0) > 0$, nondecreasing, convex, & superlinear at $+\infty$.

- Then, $\exists\, \lambda^* \in (0, +\infty)$ & $0 < \lambda < \lambda^* \Rightarrow \exists\, u_\lambda > 0$ stable classical ($L^\infty$) sol'n
  - $u_\lambda \nearrow u^*$ as $\lambda \nearrow \lambda^*$

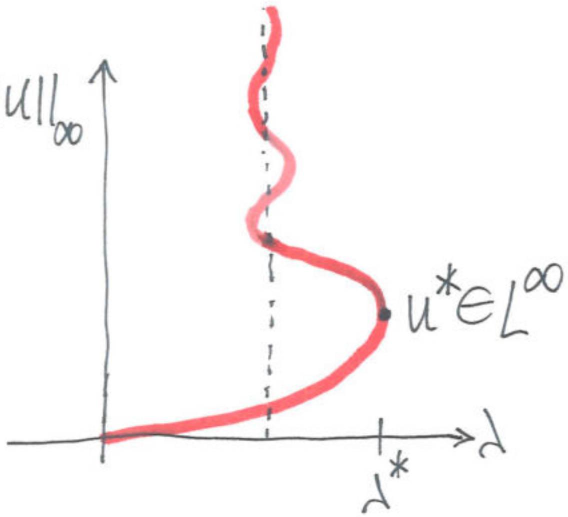
    ↳ $u^* \in L^1(\Omega)$ is a distributional stable solution for $\lambda = \lambda^*$

    $u^*$ = the extremal solution of the pb.
- $\nexists$ solutions for $\lambda > \lambda^*$

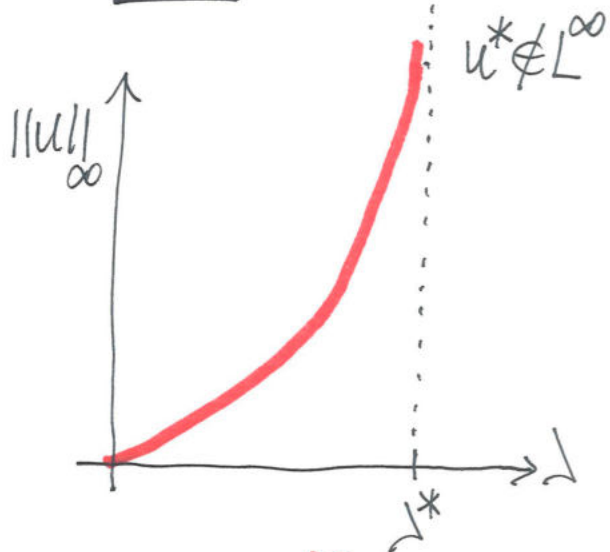
Model nonlinearities: $f(u) = e^u$ (combustion theory)

$f(u) = (1+u)^p$, $p > 1$

• [Joseph-Lundgren '72] $f(u)=e^u$ & $\Omega = B_1$ (RADIAL case) :




**$3 \leq n \leq 9$**




**$n \geq 10$**

▪ ODE techniques

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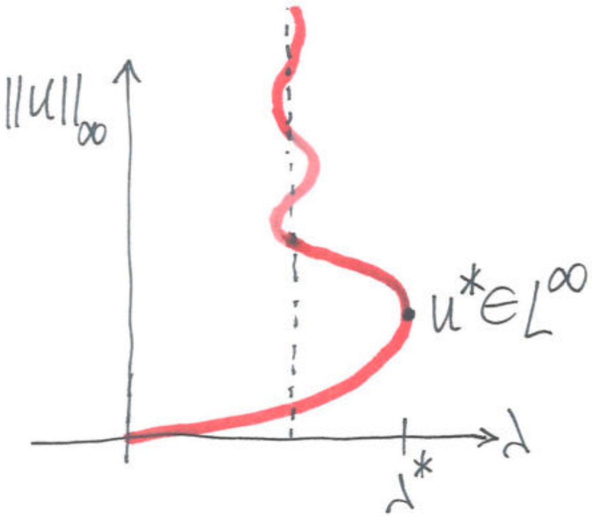


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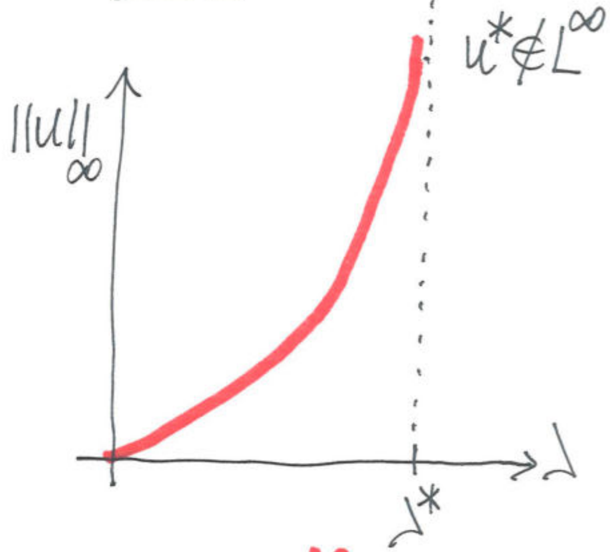


**$n \geq 10$**

▪ ODE techniques

▪ Explicit singular solution:

$u(x) = -2\log|x| \in W_0^{1,2}(B_1)$

solves $-\Delta u = 2(n-2)\,e^u$ in $B_1$, $n \geq 3$

Linearized operator $= -\Delta - 2(n-2)\frac{1}{|x|^2}$

(Hardy's ineq) $\to$ $u$ stable $\Leftrightarrow 2(n-2) \leq \frac{(n-2)^2}{4} \Leftrightarrow n \geq 10$

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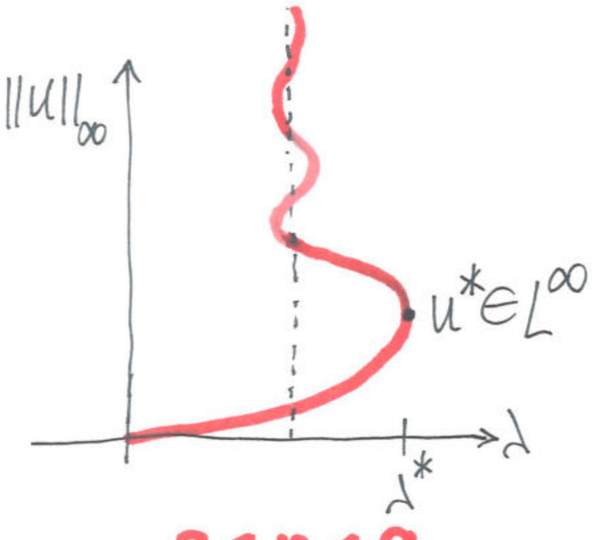


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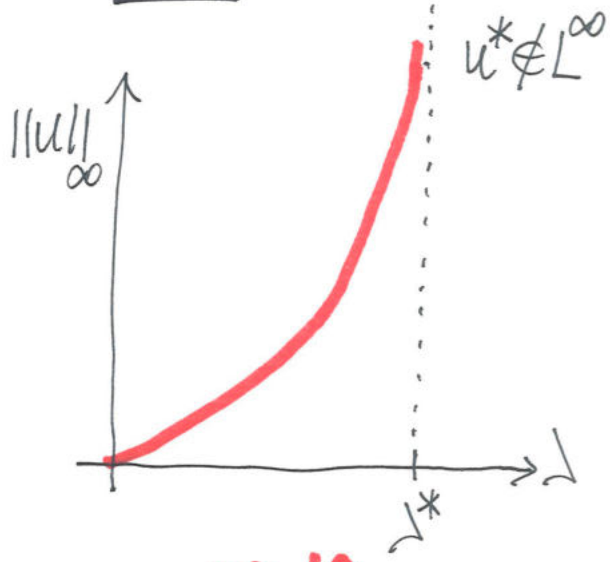


**$n \geq 10$**

- ODE techniques
- Similar for $f(u)=(1+u)^p$

$\downarrow$

explicit solutions

$u(x) = |x|^{-\alpha_p} - 1$

$(\alpha_p > 0)$

- Explicit singular solution:

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(Hardy's ineq) $\rightarrow$ $u$ stable $\Leftrightarrow$ $2(n-2) \leq \frac{(n-2)^2}{4}$ $\Leftrightarrow$ $n \geq 10$

- **Questions:** When is $\overline{u^* \in L^\infty(\Omega)}$ ?

  When are $W_0^{1,2}$ stable solutions bounded ?

- For general solutions, $L^\infty$ estimates exist for $f$ subcritical or critical : $|f(u)| \leq C(1+|u|)^p$, $p \leq \frac{n+2}{n-2}$

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- PDE analogue of "regularity of stable minimal surfaces in $\mathbb{R}^n$":
  - $\rightarrow$ Not true for $n \geq 8$
  - $\rightarrow$ True for $n=3$ ([Fischer-Colbrie & Schoen '80] [DoCarmo & Peng '79])
  - $\rightarrow$ Open pb for $4 \leq n \leq 7$ !!

($\rightarrow$ Known for $n \leq 7$ for minimizing minimal surfaces)

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  (→ Known for $n \leq 7$ for minimizing minimal surfaces)

- 1st result $\forall \Omega$ : [Crandall-Rabinowitz '75]
  $u^* \in L^\infty(\Omega)$ if $n \leq 9$ and $f(u) \sim e^u$ or $f(u) \sim (1+u)^p$

- [Brezis-Vázquez '97] Is it always $u^* \in W_0^{1,2}(\Omega)$ ?
- [Brezis '03] Is there something "sacred" about dim 10 ?
  Is it possible to construct a singular stable sol'n for $n \leq 9$, in some domain & for some $f$ ?

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- [Cabré '10] $u^* \in L^\infty(\Omega)$ if $n \leq 4$ & $\Omega$ convex
  & Interior $L^\infty$ bound if $n \leq 4$ $\forall f$ ]

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- [Villegas '13] $u^* \in L^\infty(\Omega)$ if $n \leq 4$ ; $u^* \in W_0^{1,2}(\Omega)$ if $n \leq 6$

- [Cabré & Ros-Oton '13] $L^\infty$ if $n \leq 7$ & $\Omega$ of double revolution
- [Cabré-Sanchón-Spruck '16] $L^\infty$ if $n \leq 5$ & $f'/f^{1+\varepsilon} \leq C(\varepsilon)$ $\forall \varepsilon > 0$

■ [Cabré, Figalli, Ros-Oton, Serra '19]

Thm 1 $u \in C^2(B_1)$ stable sol'n of $-\Delta u = f(u)$ in $B_1$ & $f \geq 0 \Rightarrow$

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \leq C(n) \|u\|_{L^1(B_1)} \qquad (\gamma = \gamma(n) > 0)$$

& if $n \leq 9$ then $\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C(n) \|u\|_{L^1(B_1)}$ $(\alpha = \alpha(n) > 0)$.

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**Thm 2** $\Omega$ bdd $C^3$ domain, $f \geq 0$, $f' \geq 0$, $f'' \geq 0$. $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ stable sol'n of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$ $\Rightarrow$

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \leq C(\Omega) \|u\|_{L^1(\Omega)} \qquad (\gamma = \gamma(n) > 0)$$

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Corol 2 $\Omega$ bdd $C^3$ domain $\Longrightarrow \begin{cases} u^* \in W_0^{1,2+\gamma}(\Omega) & (\gamma = \gamma(n) > 0) \\ \text{if } n \leq 9,\ u^* \in L^\infty(\Omega). \end{cases}$

Thm 3 Sharp Morrey $M^{p,\beta}(\Omega)$ estimates for stable sol'ns when $n \geq 10$.

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- PROOFS :

$$\int_\Omega f'(u)\,\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C^1_c(\Omega)$$

$$\Big\downarrow\ \xi = c \cdot \eta_{|\partial\Omega} \equiv 0$$

$$\int_\Omega c\,(\Delta c + f'(u)c)\,\eta^2 \leq \int_\Omega c^2 |\nabla \eta|^2 .$$

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$$\int_\Omega c\,(\Delta c + f'(u)c)\,\eta^2 \leq \int_\Omega c^2 |\nabla \eta|^2 .$$

[Cabré-Capella '05] motivated by Simons' lemma on minimal cones

↑

c = ‖second fund. form‖

• **Proofs** $\xi = c \cdot \eta_{|\partial\Omega} \equiv 0 \Rightarrow \int_\Omega c(\Delta c + f'(u)c)\eta^2 \leq \int_\Omega c^2 |\nabla \eta|^2.$

- [Crandall-Rabinowitz] & [Nedev] : $\underline{\xi = h(u)}$
- [Cabré-Capella] ($\Omega = B_1$) : $\xi = \underbrace{\boxed{r u_r}}_{= c} \cdot \underbrace{r^{-a} \varphi}_{= \eta}$ , $\varphi$ cut-off near $\partial B_1$

  $= x \cdot \nabla u \, |x|^{-a} \cdot \varphi$
- [Cabré '10] ($n \leq 4$) : $\xi = \underbrace{\boxed{|\nabla u|}}_{= c} \cdot \underbrace{g(u)}_{= \eta}$

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  $(n \leq 4)$

For our interior result $(n \leq 9)$ we will use both

$c = x\cdot\nabla u$

& $c = |\nabla u|$

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$(\Delta + f'(u))(x \cdot \nabla u) = 2\Delta u$

& $(\Delta + f'(u))|\nabla u| = \frac{1}{|\nabla u|}\left\{ \sum_{i,j} u_{ij}^2 - \sum_i \left( \sum_j u_{ij} \frac{u_j}{|\nabla u|} \right)^2 \right\}$

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Curvature of level sets + Michael-Simon Sobolev ineq.

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x) \rightsquigarrow \int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2$ $\rightsquigarrow$ Pohozaev trick

<u>Lemma 1</u> $\forall n\ \forall f\ \forall u$ stable sol'n $\forall \eta \in C^1_c(B_1) \Longrightarrow$

$$\int_{B_1} \{(n-2)\eta + 2x\cdot\nabla\eta\}\,\eta\,\underline{|\nabla u|^2} - 2\,\underline{(x\cdot\nabla u)}\,\underline{\nabla u\cdot\nabla(\eta^2)} \quad ①$$
$$- \underline{|x\cdot\nabla u|^2}\,|\nabla\eta|^2 \leq 0.$$

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$$\xi = (x\cdot\nabla u)\,\underbrace{|x|^{\frac{2-n}{2}}\varphi(x)}_{=\eta(x)}$$

$\frac{2-n}{2}$ so that $\{\cdots\} \geq 0$ (0)

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x) \rightsquigarrow \int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2 \rightsquigarrow$ Pohozaev trick

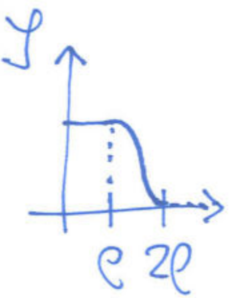
**Lemma 1** $\forall n\ \forall f\ \forall u$ stable sol'n $\forall \eta \in C^1_c(B_1) \Longrightarrow$

$$\int_{B_1} \{(n-2)\eta + 2x\cdot\nabla\eta\}\,\eta\,|\nabla u|^2 - 2(x\cdot\nabla u)\,\nabla u\cdot\nabla(\eta^2) - |x\cdot\nabla u|^2\,|\nabla\eta|^2 \le 0.$$

$$\boxed{\xi = (x\cdot\nabla u)\,\underbrace{|x|^{\frac{2-n}{2}}\varphi(x)}_{=\eta(x)}}$$

so that $\{\cdots\} \ge 0$

$$① \,\&\, ② \longrightarrow 2(n-2) - \frac{(n-2)^2}{4} = \frac{1}{4}\{8(n-2)-(n-2)^2\} = \frac{1}{4}(n-2)(10-n)$$




$$\frac{1}{4}(n-2)(10-n)\int_{B_\rho} |x|^{2-n} u_r^2 \le C \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n}|\nabla u|^2$$

$\underline{\text{NOTE}}$: $\displaystyle\int_{B_{1/2}} |x-y|^{2-n} \left|\nabla u(x)\cdot\frac{x-y}{|x-y|}\right|^2 dx \leq C \qquad \forall y \in B_{1/2}$

$\xrightarrow[\text{easy}]{}$ $\underline{u \in BMO}$ $\underline{\text{if}}$ $n \leq 9$

WE HAVE:

$$\boxed{\int_{B_\rho} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n} |\nabla u|^2}$$

NOTE: $\displaystyle\int_{B_{1/2}} |x-y|^{2-n} \left| \nabla u(x) \cdot \frac{x-y}{|x-y|} \right|^2 dx \leq C \qquad \forall y \in B_{1/2}$

$\xrightarrow[\text{easy}]{}$ $u \in BMO$ if $n \leq 9$

WE HAVE:

$$\int_{B_\rho} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho} \setminus B_\rho} |x|^{2-n} |\nabla u|^2$$

If we had $\displaystyle\int_{B_{2\rho} \setminus B_\rho} |x|^{2-n} |\nabla u|^2 \leq C' \int_{B_{2\rho} \setminus B_\rho} |x|^{2-n} u_r^2$, then

$$\to \int_{B_\rho} |x|^{2-n} u_r^2 \leq C'' \int_{B_{2\rho} \setminus B_\rho} |x|^{2-n} u_r^2$$

adimensional quantity

$$\to \int_{B_\rho} |x|^{2-n} u_r^2 \leq \frac{C''}{1+C''} \int_{B_{2\rho}} |x|^{2-n} u_r^2 \implies \text{Algebraic decay \& Hölder continuity of } u$$

We would like
$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 \leq C(n) \int_{B_{1/2}\setminus B_{1/4}} u_r^2 . \qquad (*)$$

May it be true ?

We would like

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 \leq C(n) \int_{B_{1/2}\setminus B_{1/4}} u_r^2 . \qquad (*)$$

May it be true?

If false, in the extreme case we would have

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 = 1 \quad \& \quad \int_{B_{1/2}\setminus B_{1/4}} u_r^2 = 0$$

$\rightarrow$ $u$ is 0-homogeneous

$\Downarrow$ $-\Delta u = f(u) \geq 0$

$u$ is a superharmonic f'cn on the sphere $S^{n-1}$

$u = ctt \Leftarrow$

CONTRADICTION

We would like

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 \leq C(n) \int_{B_{1/2}\setminus B_{1/4}} u_r^2 . \qquad (*)$$

May it be true ?

If false, in the extreme case we would have

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 = 1 \quad \& \quad \int_{B_{1/2}\setminus B_{1/4}} u_r^2 = 0$$

$\to$ $u$ is 0-homogeneous

$\Downarrow$ $-\Delta u = f(u) \geq 0$

$u$ is a superharmonic f'cn on the sphere $S^{n-1}$

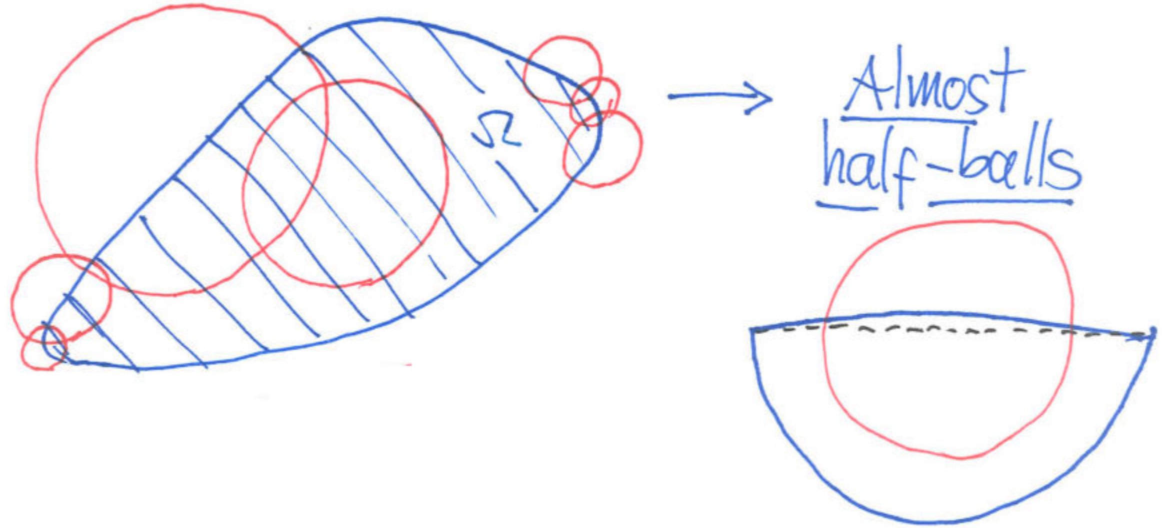
$u = ctt \Leftarrow$

CONTRADICTION

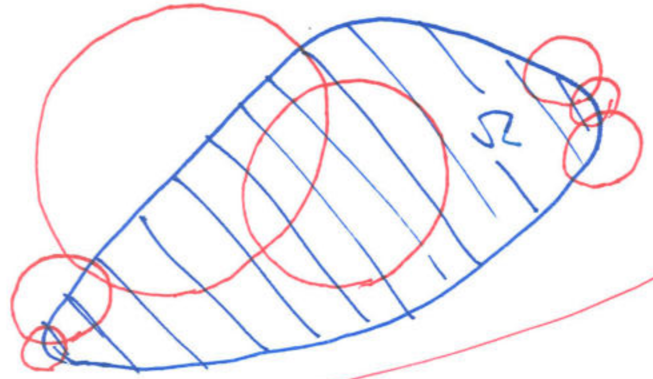
$\to$ We prove $(*)$ (under a doubling assumption that suffices) by COMPACTNESS using the higher integrability estimate

$c = |\nabla u| \Longrightarrow \|\nabla u\|_{L^{2+\gamma}} \leq C(n) \|\nabla u\|_{L^2}$

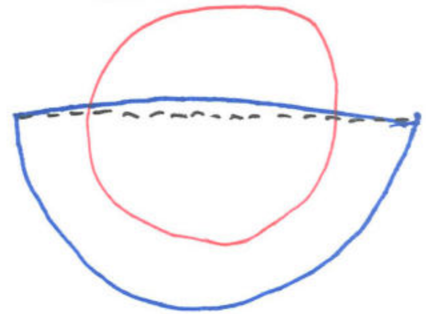
• Boundary regularity



## • Boundary regularity

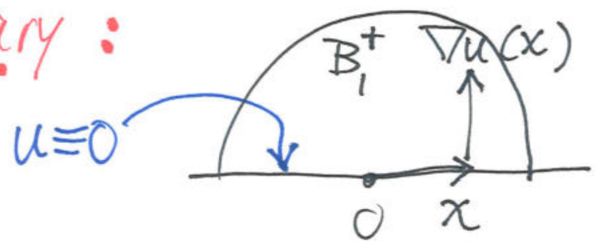



Almost half-balls



Simplest case:

Half-balls ; flat boundary :



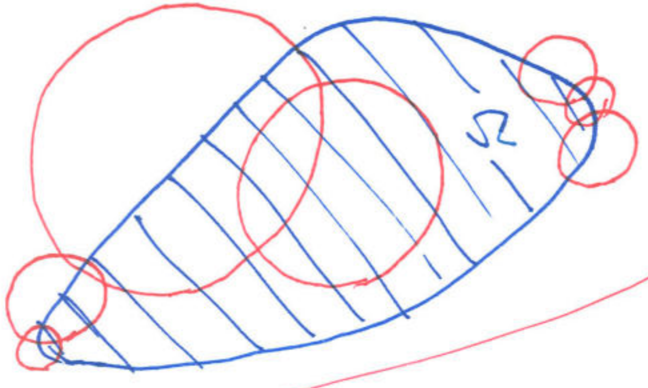

$\downarrow$

$\xi(x) = (x\cdot\nabla u)|x|^{\frac{2-n}{2}}\varphi(x)$ vanishes on the flat bdry :)

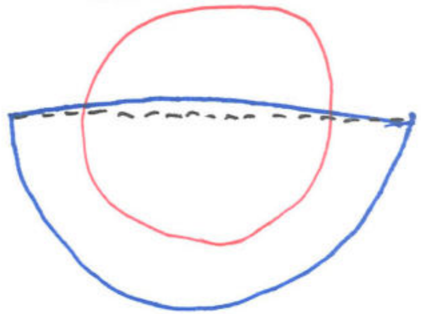
$\Downarrow$

$$(n-2)(10-n)\int_{B_\rho^+}|x|^{2-n}u_r^2 \leq C\int_{B_{2\rho}^+\setminus B_\rho^+}|x|^{2-n}|\nabla u|^2$$

- **Boundary regularity**

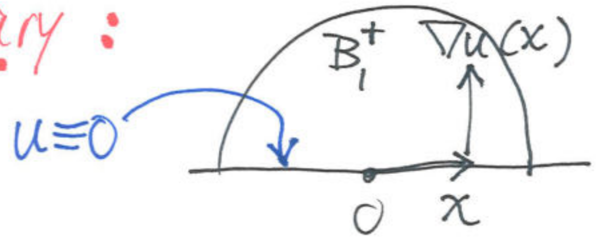



Almost half-balls



Simplest case:

**Half-balls ; flat boundary :**




$\downarrow$

$$\xi(x) = (x\cdot\nabla u)|x|^{\frac{2-n}{2}}\varphi(x)$$ vanishes on the flat bdry ☺

$$\Downarrow$$

$$(n-2)(10-n)\int_{B_\rho^+}|x|^{2-n}u_r^2 \leq C\int_{B_{2\rho}^+\setminus B_\rho^+}|x|^{2-n}|\nabla u|^2$$

We ask ¿$\exists u$, $u=0$ on $\{x_n=0\}$, $\Delta u\leq 0$ in $\{x_n>0\}$,

$$\int_{B_2^+\setminus B_1^+}|\nabla u|^2 = 1 \quad \& \quad \int_{B_2^+\setminus B_1^+}u_r^2 = 0 \ ?$$

**Yes** ☹ $u(r,\theta)=\sin\theta$

**key remark:** $u$ cannot solve $-\Delta u = f(u)$ if $u = u(\theta)$

0 homogeneous

$-2$ homogeneous

Question: Can one pass to the limit the condition $-\Delta u = f(u)$ ?

**key remark:** $u$ cannot solve $-\Delta u = f(u)$ if $u = u(\theta)$

0 homogeneous

-2 homogeneous

Question: Can one pass to the limit the condition $-\Delta u = f(u)$ ?

**Thm 4** Let $u_k$ be stable sol'ns of $-\Delta u_k = f_k(u_k)$ in $U \subset \mathbb{R}^n$ open,

with $f_k' \geq 0$, $f_k'' \geq 0$; $u_k \in W^{1,2}_{loc}(U)$

$\downarrow$ in $L^1_{loc}(U)$.

Then $u$

$u \in W^{1,2}_{loc}(U)$ is a stable solution of $-\Delta u = f(u)$ in $U$

for some $f$ nondecreasing and convex, $f: (-\infty, M) \to \mathbb{R}$.