A Polish Space of Multiparameter Persistence Modules

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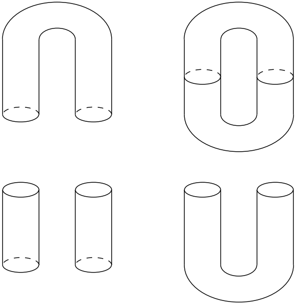
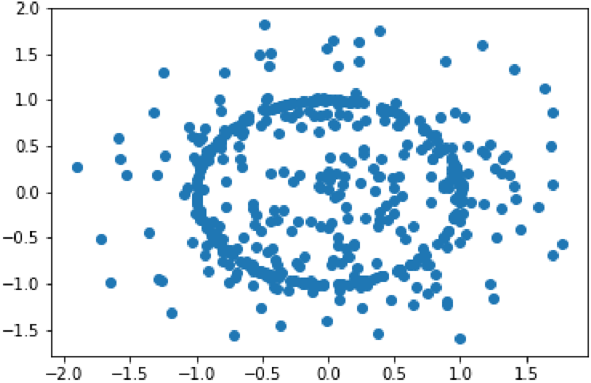
Multifiltrations



$$\begin{array}{ccc}
\mathbb{Q} & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & \mathbb{Q}^2 \\
{\scriptstyle \begin{bmatrix}1 & 1\end{bmatrix}}\Big\uparrow & & \Big\uparrow{\scriptstyle \begin{bmatrix}1\\0\end{bmatrix}} \\
\mathbb{Q}^2 & \xrightarrow[\begin{bmatrix}1 & 1\end{bmatrix}]{} & \mathbb{Q}
\end{array}$$

Examples

- Density α of a point cloud and radius r of a Rips complex around that point cloud.



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- Joint sublevel sets of a collection of functions $f_1, \cdots, f_n : X \to \mathbb{R}$, where

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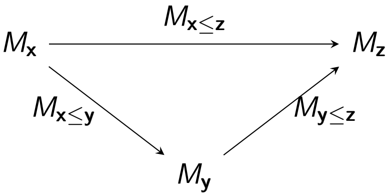
- Time t and radius r of a Rips complex on e.g. an Ising Model or a Richardson Growth model.

Definition of n-parameter persistence modules

Definition

An *n-parameter persistence module* over a field $\mathbf{k}$ is a collection of $\mathbf{k}$-vector spaces $M_{\mathbf{x}}$ for each point $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and a map $M_{\mathbf{x}\leq\mathbf{y}} : M_{\mathbf{x}} \to M_{\mathbf{y}}$ whenever $\mathbf{x} \leq \mathbf{y}$, i.e. whenever $x_k \leq y_k$ for all $k = 1, \cdots n$. We also require that whenever $\mathbf{x} \leq \mathbf{y} \leq \mathbf{z}$, we have



Definition of maps between MPMs

Definition

A map between two modules M and N are defined to be a collection of maps $\phi_{\mathbf{x}} : M_{\mathbf{x}} \to N_{\mathbf{x}}$ so that the diagram commutes.

$$\begin{array}{ccc} M_{\mathbf{y}} & \xrightarrow{\phi_{\mathbf{x}}} & N_{\mathbf{y}} \\ {\scriptstyle M_{\mathbf{x}\le\mathbf{y}}}\uparrow & & \uparrow{\scriptstyle N_{\mathbf{x}\le\mathbf{y}}} \\ M_{\mathbf{x}} & \xrightarrow[\phi_{\mathbf{y}}]{} & N_{\mathbf{x}} \end{array}$$

Shift maps

The shift functors T_ε send a module M to $T_\varepsilon M$, where

$$(T_\varepsilon M)_{\mathbf{x}} = M_{\mathbf{x}+(\varepsilon,\cdots,\varepsilon)}$$

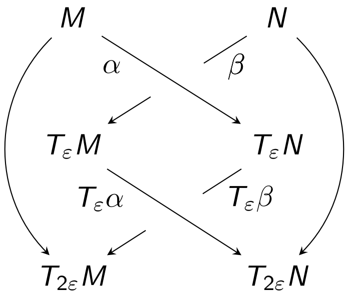
and the maps $(T_\varepsilon M)_{\mathbf{x}\leq\mathbf{y}}$ are what you'd expect. (You are literally just shifting the complex down and to left by ε).

Lemma

- *$T_0 = Id$.*
- *$T_\varepsilon T_\delta = T_{\varepsilon+\delta}$.*
- *There are natural transformations $T_\delta \implies T_\varepsilon$ for $\delta \leq \varepsilon$.*

Interleavings

We say that two modules M and N are ε-interleaved if there are maps $\alpha : M \to T_\varepsilon N$ and $\beta : N \to T_\varepsilon M$ satisfying



Definition

The *interleaving distance* between two modules M and N is

$$d_I(M,N) = \inf\{\infty\} \cup \{\varepsilon \geq 0 \mid M \text{ and } N \text{ are } \varepsilon\text{-interleaved}\}.$$

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2. Find a characterization of the (pre)compact subsets.

1-parameter Case

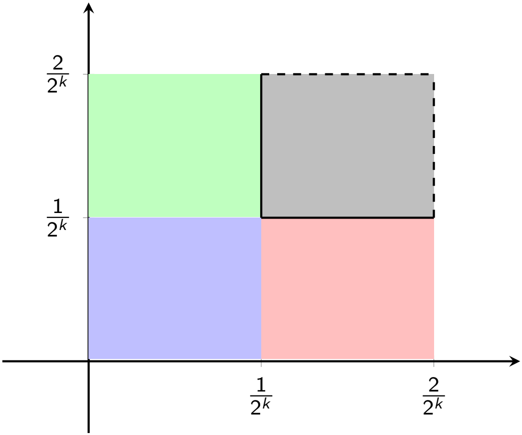
We restrict to the space of countably many finite-length barcodes which have finitely many barcodes of length $\geq \varepsilon$ for any ε. Alternatively, we restrict to the space of persistence modules M with pointwise countable dimension so that $\mathrm{Im}(M_x \to M_{x+\varepsilon})$ is finite dimensional for all ε.

Precompact subsets are the subsets $\mathcal{A}$ so that for all $\varepsilon > 0$, there exists an N_ε so that

- $\dim \mathrm{Im}(M_x \to M_{x+\varepsilon}) \leq N_\varepsilon$ for all $M \in \mathcal{A}$, and
- $\mathrm{Im}(M_x \to M_{x+\varepsilon}) = 0$ for $x \notin [-N_\varepsilon, N_\varepsilon]$.

Discrete Modules

In this talk, a discrete n-parameter persistence module with respect to the grid $2^{-k}\mathbb{Z}^n$ is a module M which is "constant in the cubes of the grid".



Mapping into Discrete Modules

We define a map P_k from to n-parameter persistence modules to the subspace of discrete modules on $2^{-k}\mathbb{Z}^n$ via

$$(P_k M)_{\mathbf{x}} = \mathrm{Im}(M_{\mathbf{x}} \to M_{\mathbf{x}+(2^{-k},\cdots,2^{-k})})$$

for $\mathbf{x} \in 2^{-k}\mathbb{Z}^n$, and complete it accordingly.

Mapping into Discrete Modules

There are two important points here:

- $d_I(M, P_k M) \leq 2^{-k}$, so the discrete modules form a dense subspace of all multiparameter persistence modules.
- If M and N are non-isomorphic discrete modules on $2^{-k}\mathbb{Z}^n$, then $d_I(M, N) \geq 2^{-k}$.

Definition of $\mathcal{C}_{\mathbf{k}}$

The subcategory of multiparameter persistence modules over a field $\mathbf{k}$ that we will be considering for the remainder of the talk is $\mathcal{C}_{\mathbf{k}}$:

Definition

$\mathcal{C}_{\mathbf{k}}$ is the full subcategory of n-parameter persistence modules over $\mathbf{k}$ where $M \in \mathcal{C}_{\mathbf{k}}$ whenever

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- for all k, there is some $N_k > 0$ so that $(P_k M)_\mathbf{x} = 0$ for $\mathbf{x} \notin [-N_k, N_k]^n$, and
- $(P_k M)_{\mathbf{x} \leq \mathbf{y}}$ is an isomorphism if $x_j = y_j$ for $j \neq k$, and x_k and y_k are not in $[-N_k, N_k]$ for some k.

Definition of $\mathcal{C}_{\mathbf{k}}$

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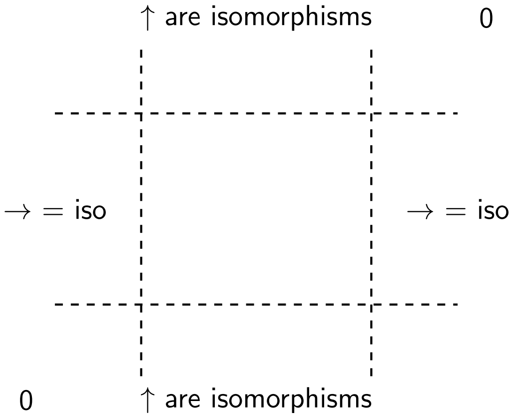
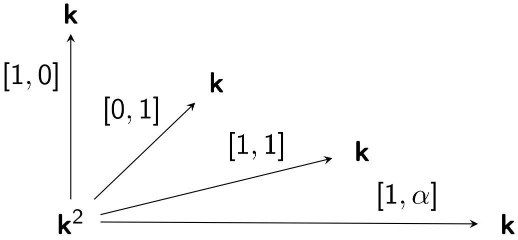


Main Result

It turns out that $\mathcal{C}_{\mathbf{k}}$ is complete, and $\mathcal{C}_{\mathbf{k}}$ is separable if and only if $\mathbf{k}$ is countable (which is optimal).

Major Example

Consider the discrete 2-parameter module $M(\alpha)$ on $\mathbb{Z}^2$ for $\alpha \in \mathbf{k}$.



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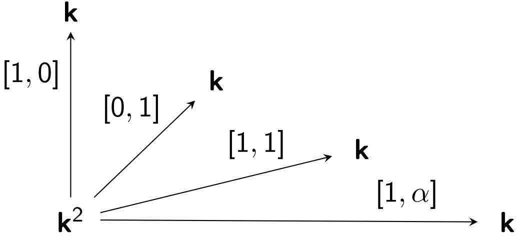

One can show that if $\alpha \neq \beta$, then $M(\alpha)$ and $M(\beta)$ are not isomorphic, so $d_I(M(\alpha), M(\beta)) \geq 1$.

The precompact subsets

A subset $\mathcal{A}$ of $\mathcal{C}_{\mathbf{k}}$ is precompact if the image of P_k restricted to $\mathcal{A}$ has finitely many isomorphism classes.

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(It's actually not quite that simple, because of the issue that even if $d_I(M, N) = 0$, we might not have that $P_k M$ and $P_k N$ are isomorphic.)

Structure on the Compact Subsets

- There is a discrete invariant for any compact subset of discrete modules on $2^{-k}\mathbb{Z}^n$.

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- There is a coarse embedding of any compact subset into some Hilbert space.

Thank you!

Metric Completeness is implied by Categorical Completeness

Theorem

A category with a flow $(\mathcal{C}, T_\varepsilon)$ is metrically complete if

- *$\mathcal{C}$ contains all categorical limits of the form $\cdots \to \bullet \to \bullet \to \bullet$, and*
- *for all $\varepsilon > 0$, T_ε preserves categorical limits.*

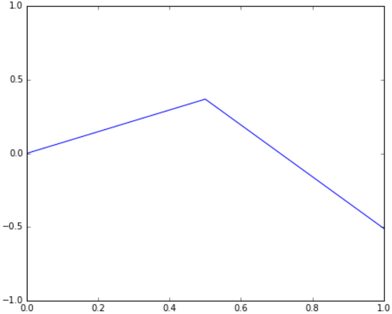
Idea of Proof

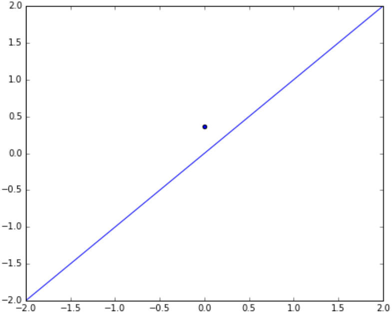
1. Start with a Cauchy sequence $\{A_k\}$ so that $d(A_k, A_{k+1})$ are $2^{-(k+1)}$-interleaved.
2. This gives a map $T_{2^{k+1}} A_{k+1} \to T_{2^k} A_k$.
3. Glue together to get a diagram

$$\cdots \to T_{\frac{1}{2^{k+1}}} A_{k+1} \to T_{\frac{1}{2^k}} A_k \to \cdots \to T_{\frac{1}{4}} A_2 \to T_{\frac{1}{2}} A_1$$

4. The (categorical) limit of this diagram turns out to be a metric limit of $\{A_k\}$.

Why we want completeness





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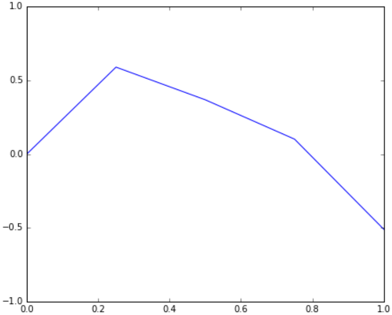

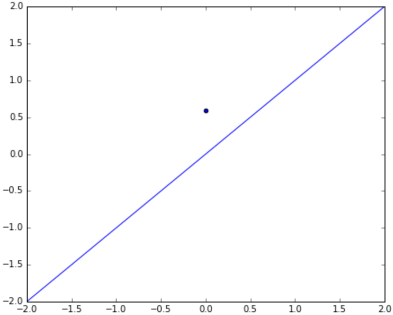

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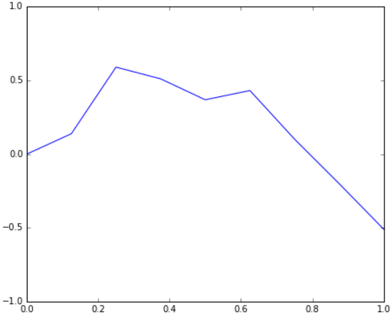

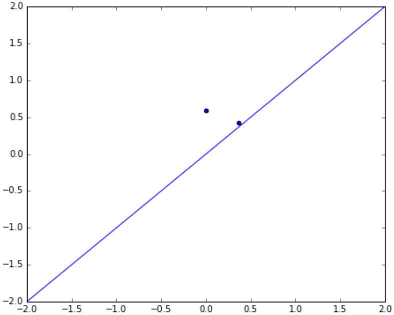

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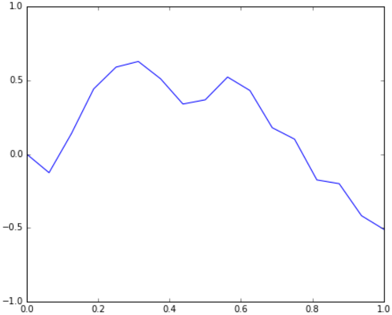

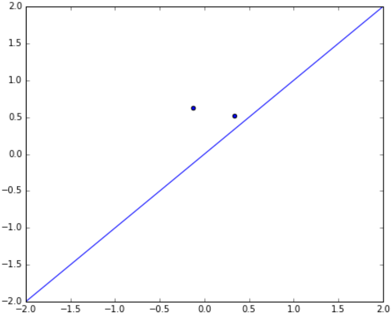

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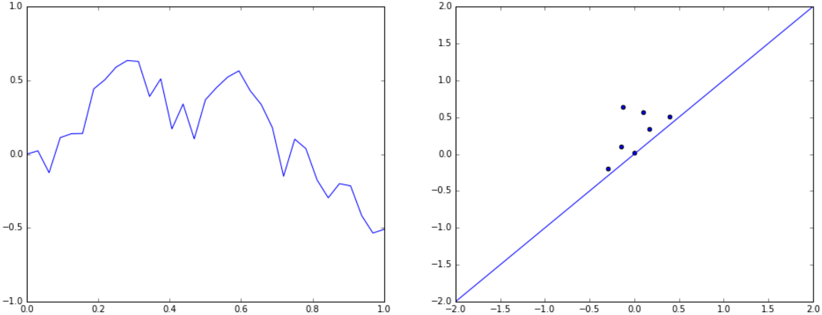

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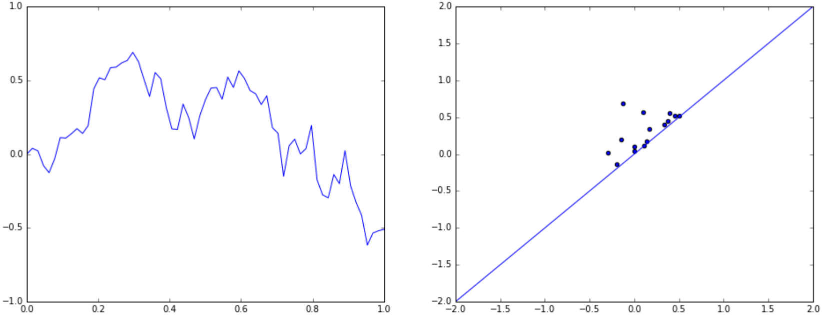

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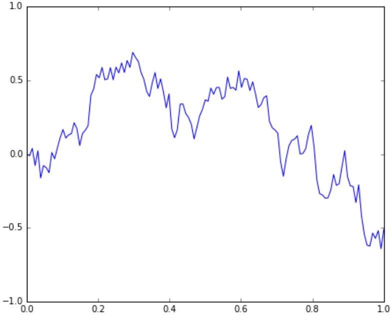

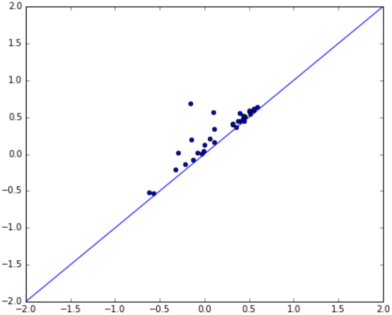

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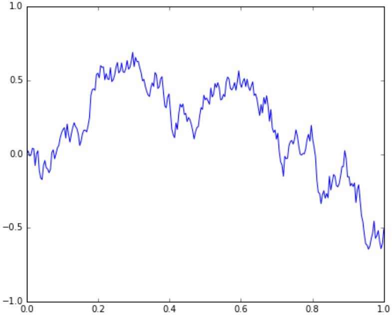

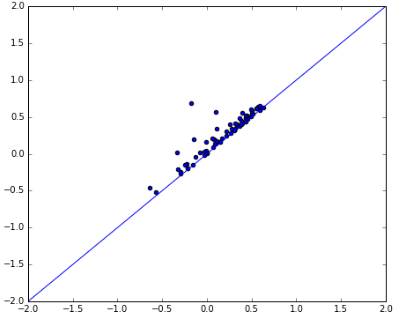

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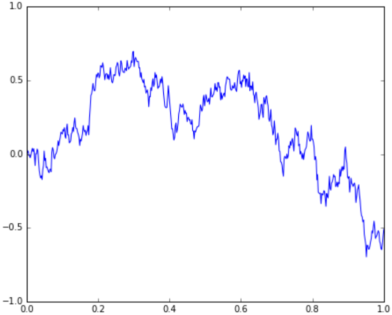

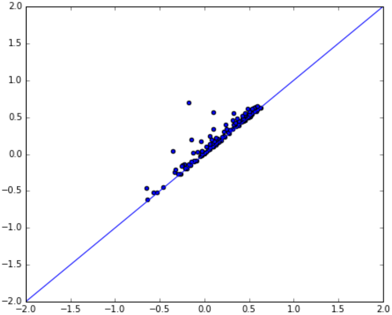

References