Universality of the Gromov–Hausdorff distance

Workshop on Topological Data Analysis
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The Fields Institute

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Goal: See that the Gromov–Hausdorff distance is a quotient of an interleaving distance, and applications of this perspective.

Outline:

- Quotient metrics, and Gromov–Hausdorff as quotient metric.
- A distance on persistent metric spaces.
- A stable bi-filtration of metric probability spaces.

Quotient metrics

$R \subseteq X \times X$ equivalence relation

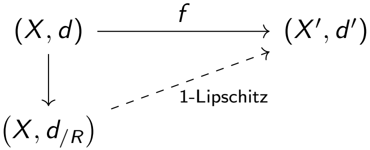
$d : X \times X \to [0, \infty]$ extended pseudo metric

Def: d is ***R*-invariant** if $xRy \Rightarrow d(x, y) = 0$.

Def: The **quotient metric** $d_{/R} : X \times X \to [0, \infty]$ is the largest R-invariant extended pseudo metric that is bounded above by d.

Universal property: A map $f : (X, d_{/R}) \to (X', d')$ is 1-Lipschitz iff

- $f : (X, d) \to (X', d')$ is 1-Lipschitz
- $xRy \Rightarrow d'(f(x), f(y)) = 0$.



Example: the Homotopy Interleaving distance

Let $X, Y \in \mathbf{Top}^{\mathbf{R}}$, $\delta \geq 0$.

Def: X and Y are **δ-interleaved** if there exist natural transformations $f : X \to Y^\delta$, $g : Y \to X^\delta$, s.t. $g^\delta \circ f =$ structure maps of $X : X \to X^{2\delta}$, $f^\delta \circ g =$ structure maps of $Y : Y \to Y^{2\delta}$. Recall: $X^\delta(r) = X(r + \delta)$.

Can think of f and g as inverse *δ-approximate natural transformations*.

Def: $d_I(X, Y) = \inf\{\delta \geq 0 \ : \ X, Y \text{ are } \delta\text{-interleaved}\}$.

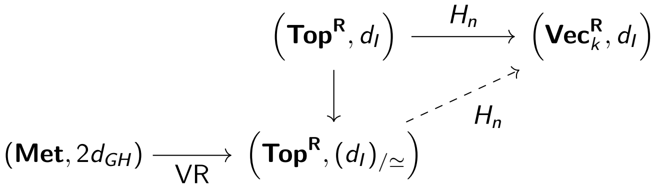
Def: $f : X \to Y$ is **weak equivalence** if all components are weak equivalences. $X \simeq Y$ if connected by zig-zag of weak equivalences.

$(d_I)_{/\simeq}$ coincides with d_{HI}, the **Homotopy Interleaving distance** of [BL].

Stability of persistent homology

Proposition (known): $H_n : \left(\mathbf{Top}^{\mathbf{R}}, d_{HI}\right) \to \left(\mathbf{Vec}_k^{\mathbf{R}}, d_I\right)$ is 1-Lipschitz.

Proof: Use UP of the Homotopy Interleaving distance.



Theorem (Blumberg, Lesnick): Vietoris–Rips is 2-Lipschitz wrt Gromov–Hausdorff and Homotopy Interleaving distances.

So any homotopy invariant yields a stable invariant of metric spaces.

Proof/Goal: Use UP of Gromov–Hausdorff distance?

Gromov–Hausdorff as a quotient interleaving distance

Let $P, Q \in \mathbf{pMet} = \{\text{pseudo metric spaces}\}$, $\delta \geq 0$.

Def: P and Q are **δ-interleaved** if $\exists$ functions $f : P \to Q$, $g : Q \to P$ that don't increase the metrics more than δ, s.t. $g \circ f = \mathrm{id}_P$, $f \circ g = \mathrm{id}_Q$.

Can think of f and g as inverse δ-*approximate* 1-*Lipschitz maps*.

Def: $d_I(P, Q) = \inf\{\delta \geq 0 \ : \ P, Q \text{ are } \delta\text{-interleaved}\}$.
Def: $f : P \to Q$ is **weak equivalence** if is surjective and distance preserving. $P \simeq Q$ if connected by a zig-zag of weak equivalences.

Theorem (S.): (Universal Property of Gromov–Hausdorff distance)

$$(d_I)_{/\simeq} = 2d_{GH}$$

Proof strategy for Rips stability: By UP of Gromov–Hausdorff distance, enough to show that VR respects interleavings and weak equivalences. . .

A rewording of proofs by Mémoli and Blumberg–Lesnick. Same argument works for other filtrations: Čech, valuation-induced filtrations [CCMSW]

The Gromov–Hausdorff interleaving distance

Let $X, Y \in \mathbf{pMet}^{\mathbf{R}}$.

Def: X and Y are **δ-interleaved** if there exist natural transformations $f : X \to Y^{\delta}$ and $g : Y \to X^{\delta}$ that don't increase metrics more than δ, s.t. $g^{\delta} \circ f =$ structure maps $: X \to X^{2\delta}$, $f^{\delta} \circ g =$ structure maps $: Y \to Y^{2\delta}$.

Def: $f : X \to Y$ is **weak equivalence** if components are surjective and distance preserving. $X \simeq Y$ if connected by zig-zag of weak equivalences.

Def: The **Gromov–Hausdorff interleaving distance** is $d_{GHI} := (d_I)_{/\simeq}$.

d_{GHI} generalizes to multi-persistent metric spaces, and recovers GH on filtered metric spaces and *slack distance* on dynamic metric spaces [KM].

Lemma: If $V : \mathbf{pMet} \to C^{\mathbf{R}}$ is functorial on δ-approximate morphisms, then $V_* : \left(\mathbf{pMet}^{\mathbf{R}^n}, d_{GHI}\right) \longrightarrow \left(C^{\mathbf{R}^{n+1}}, d_I\right)$ is Lipschitz.

This means that well-behaved, stable invariants of metric spaces yield stable invariants of multi-persistent metric spaces.

Stability of the kernel density filtration

Let $K : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a kernel (for density estimation).
Let $X \in \mathbf{cMP} = \{$compact metric probability spaces$\}$.
Def: The **kernel density filtration** of X at $s, k > 0$ is

$$\mathrm{KDF}(X)(s,k) = \left\{ x \in X \ : \ \int_{x' \in X} K\left(\frac{d_X(x,x')}{s}\right) \mathrm{d}\mu_X \geq k \right\} \subseteq X.$$

If K is uniform kernel, $\mathrm{DR}(X)(s,k) = \mathrm{VR}(\mathrm{KDF}(X)(s,k))(s)$.

Theorem (S.): $\mathrm{KDF} : \mathbf{cMP} \longrightarrow \mathbf{pMet}^{\mathbf{R}\times\mathbf{R}}$ is uniformly continuous wrt Gromov–Hausdorff–Prokhorov and Gromov–Hausdorff interleaving dist's.

Application: The persistent homology of $\mathrm{VR}(\mathrm{KDF}(X))$ is a stable invariant of metric probability spaces.

Application: If $X \in \mathbf{cMP}$ of full support, and $X_n \subseteq X$ an iid sample, then $\mathrm{KDF}(X_n) \to \mathrm{KDF}(X)$ in probability as $n \to \infty$.

The connected components of $\mathrm{VR}(\mathrm{KDF}(X))$ can be used to define stable and consistent hierarchical clustering algorithms [RS].

A. J. Blumberg and M. Lesnick
Universality of the Homotopy Interleaving Distance

S. Chowdhury, N. Clause, F. Mémoli, J. A. Sanchez, and Z. Wellner
New families of simplicial filtration functors

W. Kim and F. Mémoli
Spatiotemporal Persistent Homology for Dynamic Metric Spaces

A. Rolle and L. Scoccola
Stable and consistent density-based clustering

L. Scoccola
Locally persistent categories and metric properties of interleaving distances

Thank you for your attention!