Cup Products in Motion Planning and Coverage Problems

Clay Lecture

Fields Institute

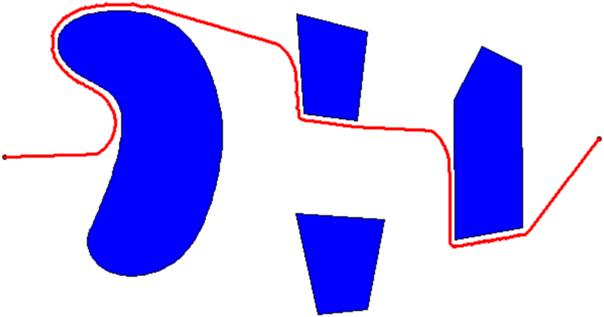
June 17, 2020

Gunnar Carlsson

aihomotopy@gmail.com

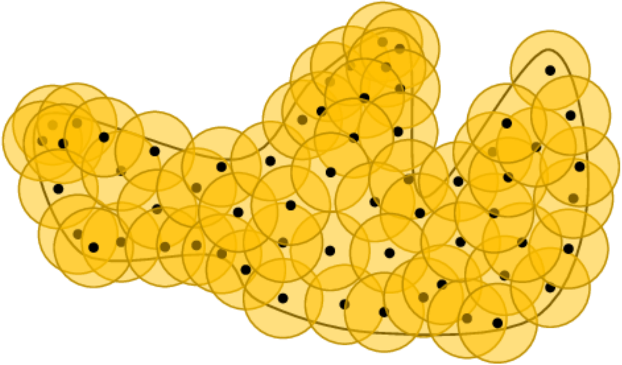
Stanford University

The Problems



Motion planning through obstacles

The Problems



Covered region for a sensor net

The Topology of Complements

Suppose we have $Y \subseteq X$ an embedding of topological spaces, and we have topological information about X and Y. What can be said about the topology of $X - Y$?

Topology of Complements

- Why should this be possible?

Topology of Complements

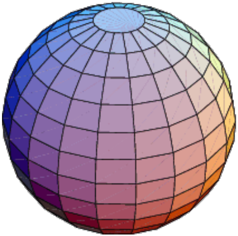
- Why should this be possible?
- First results say that we can obtain homology in the case where the ambient space X is a *manifold*

Topology of Complements

- Why should this be possible?
- First results say that we can obtain homology in the case where the ambient space X is a *manifold*
- Alexander duality theorem

Manifolds

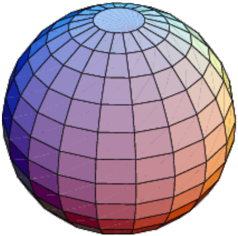
- A space X is an n-dimensional manifold if it is locally like $\mathbb{R}^n$



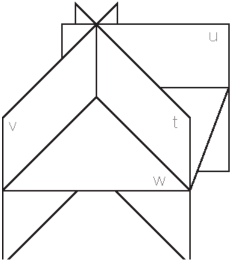
Manifold

Manifolds

- A space X is an n-dimensional manifold if it is locally like $\mathbb{R}^n$
- Means every point has a neighborhood homeomorphic to an open disc in $\mathbb{R}^n$



Manifold



Not a manifold

Cohomology

- To state theorem, requires *cohomology*

Cohomology

- To state theorem, requires *cohomology*
- Apply $Hom(-;k)$ to a chain complex, obtain a *cochain complex* with coboundary operator δ.

Cohomology

- To state theorem, requires *cohomology*
- Apply $Hom(-;k)$ to a chain complex, obtain a *cochain complex* with coboundary operator δ.
- $Ker(\delta)/im(\delta)$ defined to be cohomology $H^*(X)$.

Cohomology

- To state theorem, requires *cohomology*
- Apply $Hom(-;k)$ to a chain complex, obtain a *cochain complex* with coboundary operator δ.
- $Ker(\delta)/im(\delta)$ defined to be cohomology $H^*(X)$.
- $H^i(X) \cong H_i(X)^*$

Cohomology

- To state theorem, requires *cohomology*
- Apply $Hom(-; k)$ to a chain complex, obtain a *cochain complex* with coboundary operator δ.
- $Ker(\delta)/im(\delta)$ defined to be cohomology $H^*(X)$.
- $H^i(X) \cong H_i(X)^*$
- *Contravariant*

Alexander DualityTheorems

$Y \subseteq X$, Y compact

Alexander DualityTheorems

$Y \subseteq X$, Y compact

$X = \mathbb{R}^n$: $\tilde{H}_i(\mathbb{R}^n - Y) \cong H^{n-i-1}(Y)$

Alexander Duality Theorems

$Y \subseteq X$, Y compact

$X = \mathbb{R}^n$: $\tilde{H}_i(\mathbb{R}^n - Y) \cong H^{n-i-1}(Y)$

$X = S^n$: $\tilde{H}_i(S^n - Y) \cong \tilde{H}^{n-i-1}(Y)$

Alexander Duality Theorems

$Y \subseteq X$, Y compact

$X = \mathbb{R}^n$: $\tilde{H}_i(\mathbb{R}^n - Y) \cong H^{n-i-1}(Y)$

$X = S^n$: $\tilde{H}_i(S^n - Y) \cong \tilde{H}^{n-i-1}(Y)$

X a general manifold, $H_i(X, X - Y) \cong H^{n-i}(Y)$

Alexander Duality Theorems

This theorem tells us we can recover the homology or cohomology of complements.

Alexander Duality Theorems

This theorem tells us we can recover the homology or cohomology of complements.

Can we recover the homotopy type?

Spanier-Whitehead duality

- If a compact Y is embedded in S^n, can consider the complement of Y in $S^{n+1} \supseteq S^n$

Spanier-Whitehead duality

- If a compact Y is embedded in S^n, can consider the complement of Y in $S^{n+1} \supseteq S^n$
- $S^{n+1} - Y \cong \Sigma(S^n - Y)$

Spanier-Whitehead duality

- If a compact Y is embedded in S^n, can consider the complement of Y in $S^{n+1} \supseteq S^n$
- $S^{n+1} - Y \cong \Sigma(S^n - Y)$
- X and Y said to be *stably homotopy equivalent* if $\Sigma^i X \simeq \Sigma^i Y$ for some i

Spanier-Whitehead duality

- If a compact Y is embedded in S^n, can consider the complement of Y in $S^{n+1} \supseteq S^n$
- $S^{n+1} - Y \cong \Sigma(S^n - Y)$
- X and Y said to be *stably homotopy equivalent* if $\Sigma^i X \simeq \Sigma^i Y$ for some i
- A space determines a *stable homotopy type*

Spanier-Whitehead duality

- If a compact Y is embedded in S^n, can consider the complement of Y in $S^{n+1} \supseteq S^n$
- $S^{n+1} - Y \cong \Sigma(S^n - Y)$
- X and Y said to be *stably homotopy equivalent* if $\Sigma^i X \simeq \Sigma^i Y$ for some i
- A space determines a *stable homotopy type*
- Turns out Y determines the stable homotopy type of complement of Y for an inclusion $Y \hookrightarrow S^N$

Spanier-Whitehead duality

- If a compact Y is embedded in S^n, can consider the complement of Y in $S^{n+1} \supseteq S^n$
- $S^{n+1} - Y \cong \Sigma(S^n - Y)$
- X and Y said to be *stably homotopy equivalent* if $\Sigma^i X \simeq \Sigma^i Y$ for some i
- A space determines a *stable homotopy type*
- Turns out Y determines the stable homotopy type of complement of Y for an inclusion $Y \hookrightarrow S^N$
- Key fact is that for large N all embeddings of Y in S^N are isotopic

Spanier-Whitehead Duality

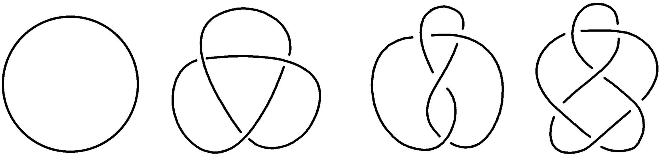
For fixed dimensions, is there actually a dependence on the embedding?

Spanier-Whitehead Duality

For fixed dimensions, is there actually a dependence on the embedding?

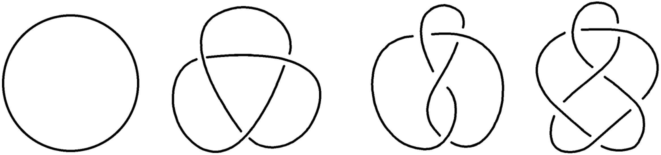
Yes

Knot Theory



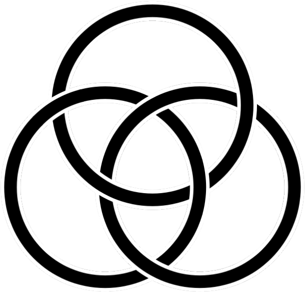
Embeddings of circle in $\mathbb{R}^3$

Knot Theory



Fundamental group of knot complement is a key invariant of a knot

Link Theory



Embeddings of disjoint union of circles in $\mathbb{R}^3$

Unstable Homotopy Types

- Homology or cohomology by itself cannot detect the difference between homotopy types of complements by the Alexander duality theorem

Unstable Homotopy Types

- Homology or cohomology by itself cannot detect the difference between homotopy types of complements by the Alexander duality theorem
- Fundamental group can in some cases

Unstable Homotopy Types

- Homology or cohomology by itself cannot detect the difference between homotopy types of complements by the Alexander duality theorem
- Fundamental group can in some cases
- Fundamental group is often complicated non-abelian group, not well suited for computation

Unstable Homotopy Types

- Homology or cohomology by itself cannot detect the difference between homotopy types of complements by the Alexander duality theorem
- Fundamental group can in some cases
- Fundamental group is often complicated non-abelian group, not well suited for computation
- Can we impose additional structure on homology or cohomology which detects unstable phenomena?

Künneth Formula

- Künneth formula describes homology and cohomology of products $X \times Y$

Künneth Formula

- Künneth formula describes homology and cohomology of products $X \times Y$
- $H_n(X \times Y) \cong \oplus_i H_{n-i}(X) \otimes H_i(Y)$

Künneth Formula

- Künneth formula describes homology and cohomology of products $X \times Y$
- $H_n(X \times Y) \cong \oplus_i H_{n-i}(X) \otimes H_i(Y)$
- $H^n(X \times Y) \cong \oplus_i H^{n-1}(X) \otimes H^i(Y)$

Künneth Formula

- Künneth formula describes homology and cohomology of products $X \times Y$
- $H_n(X \times Y) \cong \oplus_i H_{n-i}(X) \otimes H_i(Y)$
- $H^n(X \times Y) \cong \oplus_i H^{n-1}(X) \otimes H^i(Y)$
- More compact descriptions as *graded vector spaces*:

$$H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$$

and

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

Künneth Formula

- Künneth formula describes homology and cohomology of products $X \times Y$
- $H_n(X \times Y) \cong \oplus_i H_{n-i}(X) \otimes H_i(Y)$
- $H^n(X \times Y) \cong \oplus_i H^{n-1}(X) \otimes H^i(Y)$
- More compact descriptions as *graded vector spaces*:

  $$H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$$

  and

  $$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

- $\mathbb{T}$ a torus, $H_0(\mathbb{T}) = k$, $H_1(\mathbb{T}) = k^2$, and $H_2(\mathbb{T}) = k$.

Cup Products

- The cup product is a homomorphism

$$H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$$

Cup Products

- The cup product is a homomorphism

$$H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$$

- Given by the composite

$$H^*(X) \otimes H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{H^*(\Delta)} H^*(X)$$

where Δ denotes the diagonal map $X \to X \times X$, which sends x to the pair (x, x)

Cup Products

- The cup product is a homomorphism

$$H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$$

- Given by the composite

$$H^*(X) \otimes H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{H^*(\Delta)} H^*(X)$$

  where Δ denotes the diagonal map $X \to X \times X$, which sends x to the pair (x, x)
- Image of $x \otimes x'$ is denoted by $x \cup x'$

Cup Products

- The cup product is a homomorphism

$$H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$$

- Given by the composite

$$H^*(X) \otimes H^*(X) \longrightarrow H^*(X \times X) \xrightarrow{H^*(\Delta)} H^*(X)$$

  where Δ denotes the diagonal map $X \to X \times X$, which sends x to the pair (x, x)
- Image of $x \otimes x'$ is denoted by $x \cup x'$
- $H^*(X)$ becomes a graded k-algebra

Cup Products

- For X a suspension, and x and x' positive degree elements in $H^*(X)$, $x \cup x' = 0$. "Cup products of positive degree elements vanish on suspensions".

Cup Products

- For X a suspension, and x and x' positive degree elements in $H^*(X)$, $x \cup x' = 0$. "Cup products of positive degree elements vanish on suspensions".
- Means that cup products can detect difference between unstable homotopy types that are the same as stable homotopy types

Cup Products

- For X a suspension, and x and x' positive degree elements in $H^*(X)$, $x \cup x' = 0$. "Cup products of positive degree elements vanish on suspensions".
- Means that cup products can detect difference between unstable homotopy types that are the same as stable homotopy types
- Torus gives an example

Cup Products on a Torus

- The graded ring $H^*(\mathbb{T})$ is isomorphic to a *Grassmann algebra* $\Lambda(x, y)$

Cup Products on a Torus

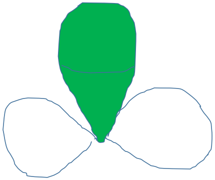
- The graded ring $H^*(\mathbb{T})$ is isomorphic to a *Grassmann algebra* $\Lambda(x, y)$
- $x \cup x = 0$, $y \cup y = 0$, and $x \cup y = -y \cup x$

Cup Products on a Torus

- The graded ring $H^*(\mathbb{T})$ is isomorphic to a *Grassmann algebra* $\Lambda(x, y)$
- $x \cup x = 0$, $y \cup y = 0$, and $x \cup y = -y \cup x$
- $x \cup y$ is non-zero, so $\mathbb{T}$ is not a suspension

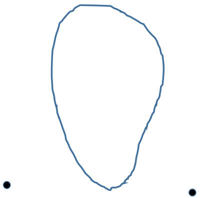
Cup Products on a Torus

- The graded ring $H^*(\mathbb{T})$ is isomorphic to a *Grassmann algebra* $\Lambda(x, y)$
- $x \cup x = 0$, $y \cup y = 0$, and $x \cup y = -y \cup x$
- $x \cup y$ is non-zero, so $\mathbb{T}$ is not a suspension
- Let $\mathcal{B}$ be the bouquet $S^1 \vee S^1 \vee S^2$



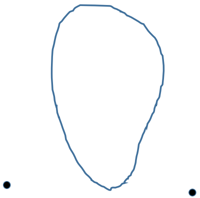
Cup Products on a Torus

- $\mathcal{B}$ is the suspension of $S^0 \vee S^0 \vee S^1$



Cup Products on a Torus

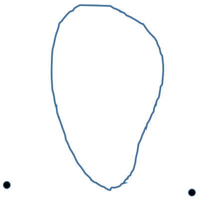
- $\mathcal{B}$ is the suspension of $S^0 \vee S^0 \vee S^1$



- Follows that cup products of positive elements vanish

Cup Products on a Torus

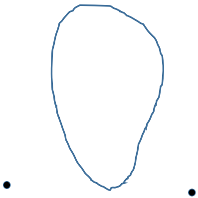
- $\mathcal{B}$ is the suspension of $S^0 \vee S^0 \vee S^1$



- Follows that cup products of positive elements vanish
- Homology and cohomology of $\mathcal{B}$ and $\mathbb{T}$ are identical as vector spaces

Cup Products on a Torus

- $\mathcal{B}$ is the suspension of $S^0 \vee S^0 \vee S^1$



- Follows that cup products of positive elements vanish
- Homology and cohomology of $\mathcal{B}$ and $\mathbb{T}$ are identical as vector spaces
- Cup product shows them to be distinct as unstable homotopy types. On the other hand, $\Sigma\mathbb{T} \simeq \Sigma\mathcal{B}$.

Cup Products and Path Components

- For a space X, $H_0(X)$ is a vector space with dimension equal to the number of path components of X.

Cup Products and Path Components

- For a space X, $H_0(X)$ is a vector space with dimension equal to the number of path components of X.
- The path components determines a basis for the vector space $H_0(X)$

Cup Products and Path Components

- For a space X, $H_0(X)$ is a vector space with dimension equal to the number of path components of X.
- The path components determines a basis for the vector space $H_0(X)$
- Homology alone does not permit us to identify the basis

Cup Products and Path Components

- For a space X, $H_0(X)$ is a vector space with dimension equal to the number of path components of X.
- The path components determines a basis for the vector space $H_0(X)$
- Homology alone does not permit us to identify the basis
- $H_0(-)$ does not determine $\pi_0(-)$ *as a set-valued functor*

Cup Products and Path Components

- $H^0(X)$ is a ring under cup product

Cup Products and Path Components

- $H^0(X)$ is a ring under cup product
- $H^0(X)$ is isomorphic to the k-algebra of k-valued functions on $\pi_0(X)$ under pointwise addition and multiplication

Cup Products and Path Components

- $H^0(X)$ is a ring under cup product
- $H^0(X)$ is isomorphic to the k-algebra of k-valued functions on $\pi_0(X)$ under pointwise addition and multiplication
- The set of k-algebra homomorphisms $H^0(X) \to k$ is in one to one correspondence with the elements of $\pi_0(X)$

Cup Products and Path Components

- $H^0(X)$ is a ring under cup product
- $H^0(X)$ is isomorphic to the k-algebra of k-valued functions on $\pi_0(X)$ under pointwise addition and multiplication
- The set of k-algebra homomorphisms $H^0(X) \to k$ is in one to one correspondence with the elements of $\pi_0(X)$
- Means that we *can* recover π_0 from the k-algebra valued functor $H^0(-)$

Cup Products of Complements

- The various duality theorems allow us to understand the homology of the complements, including H_0.

Cup Products of Complements

- The various duality theorems allow us to understand the homology of the complements, including H_0.
- Is it possible to use the same methods to recover cup products of the complement, and consequently the set π_0 applied to the complement?

Cup Products of Complements

- The various duality theorems allow us to understand the homology of the complements, including H_0.
- Is it possible to use the same methods to recover cup products of the complement, and consequently the set π_0 applied to the complement?
- This can be done, using the fact that cup products are induced by a map, namely the diagonal map, and a functoriality result for the duality theorems

Functoriality of Alexander Duality

All the Alexander Duality isomorphism for $X = S^n$ described above is functorial, in the sense that for an inclusion $Y_0 \subseteq Y_1 \subseteq X$, the diagram

$$\begin{array}{ccc} H^i(X - Y_0) & \longrightarrow & H_{n-i-1}(Y_0) \\ \downarrow & & \downarrow \\ H^i(X - Y_1) & \longrightarrow & H_{n-i-1}(Y_1) \end{array}$$

commutes. Note that we have $X - Y_1 \hookrightarrow X - Y_0$. There are analogous statements for $X = \mathbb{R}^n$ or more general.

Cup Products of Complements

- We assume that we are given a compact subset $A \subseteq \mathbb{R}^n$, and let $CA = \mathbb{R}^n - A$

Cup Products of Complements

- We assume that we are given a compact subset $A \subseteq \mathbb{R}^n$, and let $CA = \mathbb{R}^n - A$
- $\tilde{H}_i(CA) \cong H^{n-i-1}(A)$ by Alexander Duality Theorem

Cup Products of Complements

- We assume that we are given a compact subset $A \subseteq \mathbb{R}^n$, and let $CA = \mathbb{R}^n - A$
- $\tilde{H}_i(CA) \cong H^{n-i-1}(A)$ by Alexander Duality Theorem
- Let $C_\Delta A$ denote the complement of $A \subseteq A \times A \to C_\Delta A$

Cup Products of Complements

- We assume that we are given a compact subset $A \subseteq \mathbb{R}^n$, and let $CA = \mathbb{R}^n - A$
- $\tilde{H}_i(CA) \cong H^{n-i-1}(A)$ by Alexander Duality Theorem
- Let $C_\Delta A$ denote the complement of $A \subseteq A \times A \to C_\Delta A$
- Let $C(A \times A)$ denote the complement of $A \times A$ in $\mathbb{R}^{2n}$

Cup Products of Complements

- We assume that we are given a compact subset $A \subseteq \mathbb{R}^n$, and let $CA = \mathbb{R}^n - A$
- $\tilde{H}_i(CA) \cong H^{n-i-1}(A)$ by Alexander Duality Theorem
- Let $C_\Delta A$ denote the complement of $A \subseteq A \times A \to C_\Delta A$
- Let $C(A \times A)$ denote the complement of $A \times A$ in $\mathbb{R}^{2n}$
- $H^0(A \times A) \cong \tilde{H}_{2n-1}(C(A \times A))$ and $H^0(A) \cong \tilde{H}_{2n-1}(C_\Delta A)$

Cup Products of Complements

- We assume that we are given a compact subset $A \subseteq \mathbb{R}^n$, and let $CA = \mathbb{R}^n - A$
- $\tilde{H}_i(CA) \cong H^{n-i-1}(A)$ by Alexander Duality Theorem
- Let $C_\Delta A$ denote the complement of $A \subseteq A \times A \to C_\Delta A$
- Let $C(A \times A)$ denote the complement of $A \times A$ in $\mathbb{R}^{2n}$
- $H^0(A \times A) \cong \tilde{H}_{2n-1}(C(A \times A))$ and $H^0(A) \cong \tilde{H}_{2n-1}(C_\Delta A)$
- Obtain map $C(A \times A) \to C_\Delta A$ inducing the cup product on $H^0(A)$ via functoriality of Alexander Duality

The Adams Spectral Sequence

- How to go beyond π_0?

The Adams Spectral Sequence

- How to go beyond π_0?
- Exists a methodology called the (unstable) Adams Spectral Sequence, due to Bousfiled-Kan

The Adams Spectral Sequence

- How to go beyond π_0?
- Exists a methodology called the (unstable) Adams Spectral Sequence, due to Bousfiled-Kan
- Depends on cohomology as an algebra, as well as *cohomology operations*

The Adams Spectral Sequence

- How to go beyond π_0?
- Exists a methodology called the (unstable) Adams Spectral Sequence, due to Bousfiled-Kan
- Depends on cohomology as an algebra, as well as *cohomology operations*
- Produces an output which gives a great deal of information about the higher dimensional homotopy

The Adams Spectral Sequence

- How to go beyond π_0?
- Exists a methodology called the (unstable) Adams Spectral Sequence, due to Bousfiled-Kan
- Depends on cohomology as an algebra, as well as *cohomology operations*
- Produces an output which gives a great deal of information about the higher dimensional homotopy
- Complete up to “extension problems” and “differentials”

Time Varying Robotics and Sensor Net Problems

- We may have a motion planning problem or a situation in which we have moving sensors

Time Varying Robotics and Sensor Net Problems

- We may have a motion planning problem or a situation in which we have moving sensors
- In this case we want to have path in the ambient space which avoids the obstacles at every point in time

Time Varying Robotics and Sensor Net Problems

- We may have a motion planning problem or a situation in which we have moving sensors
- In this case we want to have path in the ambient space which avoids the obstacles at every point in time
- Formulate the problem in terms of *spaces over a base*

Time Varying Robotics and Sensor Net Problems

- We may have a motion planning problem or a situation in which we have moving sensors
- In this case we want to have path in the ambient space which avoids the obstacles at every point in time
- Formulate the problem in terms of *spaces over a base*
- Ambient space will now be $X = [0,1] \times \mathbb{R}^n$, consisting of a position and a time

Time Varying Robotics and Sensor Net Problems

- We may have a motion planning problem or a situation in which we have moving sensors
- In this case we want to have path in the ambient space which avoids the obstacles at every point in time
- Formulate the problem in terms of *spaces over a base*
- Ambient space will now be $X = [0,1] \times \mathbb{R}^n$, consisting of a position and a time
- $\pi : X \to [0,1]$ is the projection

Time Varying Robotics and Sensor Net Problems

- We may have a motion planning problem or a situation in which we have moving sensors
- In this case we want to have path in the ambient space which avoids the obstacles at every point in time
- Formulate the problem in terms of *spaces over a base*
- Ambient space will now be $X = [0, 1] \times \mathbb{R}^n$, consisting of a position and a time
- $\pi : X \to [0, 1]$ is the projection
- The time varying obstacles will now consist of a subspace $Y \subseteq X$

Time Varying Robotics and Sensor Net Problems

- By an *admissible path*, we'll mean a section

$$\sigma : [0, 1] \to X - Y$$

i.e. a continuous map $\sigma : [0, 1] \to Y$ so that $\pi \circ s = id_{[0,1]}$

Time Varying Robotics and Sensor Net Problems

- By an *admissible path*, we'll mean a section

$$\sigma : [0,1] \to X - Y$$

i.e. a continuous map $\sigma : [0,1] \to Y$ so that $\pi \circ s = id_{[0,1]}$

- Many questions can be asked about such sections

Time Varying Robotics and Sensor Net Problems

- By an *admissible path*, we'll mean a section

$$\sigma : [0,1] \to X - Y$$

  i.e. a continuous map $\sigma : [0,1] \to Y$ so that $\pi \circ s = id_{[0,1]}$
- Many questions can be asked about such sections
- Does one exist?

Time Varying Robotics and Sensor Net Problems

- By an *admissible path*, we'll mean a section

$$\sigma : [0,1] \to X - Y$$

  i.e. a continuous map $\sigma : [0,1] \to Y$ so that $\pi \circ s = id_{[0,1]}$
- Many questions can be asked about such sections
- Does one exist?
- How many homotopy classes are there?

Time Varying Robotics and Sensor Net Problems

- By an *admissible path*, we'll mean a section

$$\sigma : [0,1] \to X - Y$$

i.e. a continuous map $\sigma : [0,1] \to Y$ so that $\pi \circ s = id_{[0,1]}$

- Many questions can be asked about such sections
- Does one exist?
- How many homotopy classes are there?
- What is the structure of the space of sections?

Time Varying Robotics and Sensor Net Problems

- Interesting work done on the first problem by Ghrist-de Silva, Adams-C., and Ghrist-Krishnan

Time Varying Robotics and Sensor Net Problems

- Interesting work done on the first problem by Ghrist-de Silva, Adams-C., and Ghrist-Krishnan
- Second and third problems not addressed

Time Varying Robotics and Sensor Net Problems

- Interesting work done on the first problem by Ghrist-de Silva, Adams-C., and Ghrist-Krishnan
- Second and third problems not addressed
- Second and third problems potentially useful as starting points for finding optimal paths

Time Varying Robotics and Sensor Net Problems

- Interesting work done on the first problem by Ghrist-de Silva, Adams-C., and Ghrist-Krishnan
- Second and third problems not addressed
- Second and third problems potentially useful as starting points for finding optimal paths
- Will describe new work in this direction joint with B. Filippenko

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach

- Zig-zag diagram in a category is a diagram with shape

$$X_0 \leftarrow Y_0 \rightarrow X_1 \leftarrow Y_1 \rightarrow X_2 \leftarrow Y_2$$

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach

- Zig-zag diagram in a category is a diagram with shape

$$X_0 \leftarrow Y_0 \rightarrow X_1 \leftarrow Y_1 \rightarrow X_2 \leftarrow Y_2$$

- Zig-zag diagrams of vector spaces also have a classification theorem in terms of barcodes

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach

- Zig-zag diagram in a category is a diagram with shape

$$X_0 \leftarrow Y_0 \rightarrow X_1 \leftarrow Y_1 \rightarrow X_2 \leftarrow Y_2$$

- Zig-zag diagrams of vector spaces also have a classification theorem in terms of barcodes
- Irreducible summands are of the form

$$0 \rightarrow 0 \leftarrow k \rightarrow k \leftarrow k \rightarrow 0 \leftarrow 0$$

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach

- Zig-zag diagram in a category is a diagram with shape

$$X_0 \leftarrow Y_0 \rightarrow X_1 \leftarrow Y_1 \rightarrow X_2 \leftarrow Y_2$$

- Zig-zag diagrams of vector spaces also have a classification theorem in terms of barcodes
- Irreducible summands are of the form

$$0 \rightarrow 0 \leftarrow k \rightarrow k \leftarrow k \rightarrow 0 \leftarrow 0$$

- Theorem due to P. Gabriel

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach

- Break up $[0, 1]$ into closed intervals $I_0, I_1, \ldots, I_n$ that intersect in endpoints only

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach

- Break up $[0, 1]$ into closed intervals $I_0, I_1, \ldots, I_n$ that intersect in endpoints only
- Construct the spaces $\pi^{-1}(I_s)$ and $\pi^{-1}(I_s \cap I_{s+1})$, and create a zig-zag diagram of spaces

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach

- Break up $[0,1]$ into closed intervals $I_0, I_1, \ldots, I_n$ that intersect in endpoints only
- Construct the spaces $\pi^{-1}(I_s)$ and $\pi^{-1}(I_s \cap I_{s+1})$, and create a zig-zag diagram of spaces
- Clear that if there is a section, then zig-zag barcode for H_0 will have a long bar, since the constant zig-zag

$$k \leftarrow k \rightarrow k \leftarrow k \rightarrow k \leftarrow k$$

of dimension 1 is contained as a summand

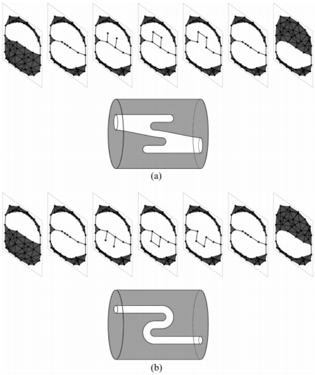
Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach

- Break up $[0, 1]$ into closed intervals $I_0, I_1, \ldots, I_n$ that intersect in endpoints only
- Construct the spaces $\pi^{-1}(I_s)$ and $\pi^{-1}(I_s \cap I_{s+1})$, and create a zig-zag diagram of spaces
- Clear that if there is a section, then zig-zag barcode for H_0 will have a long bar, since the constant zig-zag

$$k \leftarrow k \rightarrow k \leftarrow k \rightarrow k \leftarrow k$$

  of dimension 1 is contained as a summand
- Converse is *not* true

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach



Analogue of two distinct knots in this setting

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Consider the idealized case where the covered region Y is a codimension zero submanifold with boundary, and where the projection π is a Morse function when restricted to ∂Y

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Consider the idealized case where the covered region Y is a codimension zero submanifold with boundary, and where the projection π is a Morse function when restricted to ∂Y
- Use the finite set of critical values in $[0, 1]$ to build a zig-zag

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Consider the idealized case where the covered region Y is a codimension zero submanifold with boundary, and where the projection π is a Morse function when restricted to ∂Y
- Use the finite set of critical values in $[0,1]$ to build a zig-zag
- We are able to perform duality over the base $[0,1]$ to obtain cup product information concerning the zig-zag of algebras for $H^0(X - Y)$, where X is $\mathbb{R}^n \times [0,1]$ (result is actually more general)

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Consider the idealized case where the covered region Y is a codimension zero submanifold with boundary, and where the projection π is a Morse function when restricted to ∂Y
- Use the finite set of critical values in $[0,1]$ to build a zig-zag
- We are able to perform duality over the base $[0,1]$ to obtain cup product information concerning the zig-zag of algebras for $H^0(X - Y)$, where X is $\mathbb{R}^n \times [0,1]$ (result is actually more general)
- This in turn allows us to compute the zig-zag of sets $\pi_0(X - Y)$

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Study the zig-zag of sets $\pi_0(X - Y)$ to obtain a zig-zag of sets of connected components

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Study the zig-zag of sets $\pi_0(X - Y)$ to obtain a zig-zag of sets of connected components
- Let Γ denote the space of sections of the space $X - Y \longrightarrow [0, 1]$

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Study the zig-zag of sets $\pi_0(X - Y)$ to obtain a zig-zag of sets of connected components
- Let Γ denote the space of sections of the space $X - Y \longrightarrow [0,1]$
- We prove that there is a surjective map

$$\pi_0\Gamma \longrightarrow \varprojlim \pi_0(X - Y)$$

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Study the zig-zag of sets $\pi_0(X - Y)$ to obtain a zig-zag of sets of connected components
- Let Γ denote the space of sections of the space $X - Y \longrightarrow [0, 1]$
- We prove that there is a surjective map

$$\pi_0\Gamma \longrightarrow \varprojlim \pi_0(X - Y)$$

- Need to analyze fibers

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Given an element $\xi \in \varprojlim \pi_0(X - Y)$, we obtain the group $G = \prod \pi_1(X_i, \xi_i)$, where the product is over the critical values and X_i is the inverse image of the given critical value

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Given an element $\xi \in \varprojlim \pi_0(X - Y)$, we obtain the group $G = \prod \pi_1(X_i, \xi_i)$, where the product is over the critical values and X_i is the inverse image of the given critical value
- Let W denote the set $H = \prod \pi_1(X(i, i+1))$, where $X(i, i+1)$ is the set of points whose value lies between the i-th and the $(i+1)$-th critical value

Time Varying Robotics and Sensor Net Problems: Zig-Zag Approach (w. B. Filippenko)

- Given an element $\xi \in \varprojlim \pi_0(X - Y)$, we obtain the group $G = \prod \pi_1(X_i, \xi_i)$, where the product is over the critical values and X_i is the inverse image of the given critical value
- Let W denote the set $H = \prod \pi_1(X(i, i+1))$, where $X(i, i+1)$ is the set of points whose value lies between the i-th and the $(i+1)$-th critical value
- The group has a natural action on the set H, and the orbit set G/H is in bijective correspondence with the inverse image of ξ in Γ

Future Work

- Develop methods obtaining information in sampled situation - versions of Niyogi-Smale Weinberger of importance here

Future Work

- Develop methods obtaining information in sampled situation - versions of Niyogi-Smale Weinberger of importance here
- Develop parametrized (over a base) unstable Adams spectral sequence to obtain geometric information from algebraic invariants (Wyatt Mackey)

Future Work

- Develop methods obtaining information in sampled situation - versions of Niyogi-Smale Weinberger of importance here
- Develop parametrized (over a base) unstable Adams spectral sequence to obtain geometric information from algebraic invariants (Wyatt Mackey)
- Develop embedding calculus over $[0, 1]$ to obtain unstable information about complement. Pursued by G. Arone and A. Jin

Future Work

- Develop methods obtaining information in sampled situation - versions of Niyogi-Smale Weinberger of importance here
- Develop parametrized (over a base) unstable Adams spectral sequence to obtain geometric information from algebraic invariants (Wyatt Mackey)
- Develop embedding calculus over $[0, 1]$ to obtain unstable information about complement. Pursued by G. Arone and A. Jin
- In motion planning problems, use these calculations to yield starting points for optimization algorithms for shortest paths

Future Work

- Develop methods obtaining information in sampled situation - versions of Niyogi-Smale Weinberger of importance here
- Develop parametrized (over a base) unstable Adams spectral sequence to obtain geometric information from algebraic invariants (Wyatt Mackey)
- Develop embedding calculus over $[0, 1]$ to obtain unstable information about complement. Pursued by G. Arone and A. Jin
- In motion planning problems, use these calculations to yield starting points for optimization algorithms for shortest paths
- "Sheafify" whole construction so as not to rely on choices of zig-zags

Thank You!