Persistent homotopy theory

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June 17, 2020

Basic setup

X finite, $X \subset Z$, Z a metric space.

• $P_s(X)$ = poset of subsets $\sigma \subset X$ such that $d(x,y) \leq s$ for all $x, y \in \sigma$.

$P_s(X)$ is the poset of non-degenerate simplices of the Vietoris-Rips complex $V_s(X)$. $BP_s(X)$ is **barycentric subdivision** of $V_s(X)$.

We have poset inclusions

$$\sigma : P_s(X) \subset P_t(X), \; s \leq t,$$

$P_0(X) = X$, and $P_t(X) = \mathcal{P}(X)$ (all subsets of X) for t suff large.

• $k \geq 0$: $P_{s,k}(X) \subset P_s(X)$ subposet of simplices σ such that each element $x \in \sigma$ has at least k neighbours y such that $d(x,y) \leq s$.

$P_{s,k}(X)$ is the poset of non-degenerate simplices of the degree Rips complex $L_{s,k}(X)$, again the **barycentric subdivision**.

Stability results

Theorem 1 (Rips stability).

Suppose $X \subset Y$ in $D(Z)$ such that $d_H(X,Y) < r$. There is a homotopy commutative diagram (homotopy interleaving)

$$\begin{array}{ccc} P_s(X) & \xrightarrow{\sigma} & P_{s+2r}(X) \\ i\downarrow & \overset{\theta}{\nearrow} & \downarrow i \\ P_s(Y) & \xrightarrow[\sigma]{} & P_{s+2r}(Y) \end{array}$$

Theorem 2.

Suppose $X \subset Y$ in $D(Z)$ such that $d_H(X^{k+1}_{dis}, Y^{k+1}_{dis}) < r$. There is a homotopy commutative diagram

$$\begin{array}{ccc} P_{s,k}(X) & \xrightarrow{\sigma} & P_{s+2r,k}(X) \\ i\downarrow & \overset{\theta}{\nearrow} & \downarrow i \\ P_{s,k}(Y) & \xrightarrow[\sigma]{} & P_{s+2r,k}(Y) \end{array}$$

Controlled equivalences

NB: $V_*(X) := BP_*(X)$ henceforth.

Suppose that $X \subset Y$ in $D(Z)$ and we have a homotopy interleaving

$$\begin{array}{ccc} V_s(X) & \xrightarrow{\sigma} & V_{s+2r}(X) \\ i\downarrow & \overset{\theta}{\nearrow} & \downarrow i \\ V_s(Y) & \xrightarrow[\sigma]{} & V_{s+2r}(Y) \end{array}$$

1) $i : \pi_0 V_*(X) \to \pi_0 V_*(Y)$ is a $2r$-**monomorphism**: if $i([x]) = i([y])$ in $\pi_0 V_s(Y)$ then $\sigma[x] = \sigma[y]$ in $\pi_0 V_{s+2r}(X)$

2) $i : \pi_0 V_*(X) \to \pi_0 V_*(Y)$ is a $2r$-**epimorphism**: given $[y] \in \pi_0 V_s(Y)$, $\sigma[y] = i[x]$ for some $[x] \in \pi_0 V_{s+2r}(X)$.

3) All $i : \pi_n(V_*(X), x) \to \pi_n(V_*(Y), i(x))$ are $2r$-**isomorphisms**.

The map $i : V_*(X) \to V_*(Y)$ is a $2r$-**equivalence** of systems.

Systems of spaces

A system of spaces is a functor $X : [0, \infty) \to s\mathrm{Set}$, aka. a diagram of simplicial sets with index category $[0, \infty)$.

A map of systems $X \to Y$ is a natural transformation of functors defined on $[0, \infty)$.

Examples

1) The functor $V_*(X)$, $s \mapsto V_s(X) = BP_s(X)$ is a system of spaces, for a data set $X \subset Z$.

2) If $X \subset Y \subset Z$ are data sets, the induced maps $P_s(X) \to P_s(Y)$, $V_s(X) \to V_s(Y)$ define maps of systems $P_*(X) \to P_*(Y)$ (posets) and $V_*(X) \to V_*(Y)$ (spaces).

Homotopy types

There are many ways to discuss homotopy types of systems. The oldest is the **projective structure** (Bousfield-Kan, 1972):

A map $f : X \to Y$ is a **weak equivalence** (resp. **fibration**) if each map $X_s \to Y_s$ is a weak equiv. (resp. fibration) of simplicial sets.

A map $A \to B$ is a **projective cofibration** if it has the left lifting property with respect all maps which are trivial fibrations.

Lemma 3.

Suppose that $X \subset Y \subset Z$ are data sets. Then $V_*(X) \to V_*(Y)$ is a projective cofibration.

The map $V_*(X) \to V_*(Y)$ is also a sectionwise cofibration, i.e. all maps $V_s(X) \to V_s(Y)$ are monomorphisms.

r-equivalences

Suppose that $f : X \to Y$ is a map of systems. Say that f is an r-**equivalence** if

1) the map $f : \pi_0(X) \to \pi_0(Y)$ is an r-isomorphism of systems of sets
2) the maps $f : \pi_k(X_s, x) \to \pi_k(Y_s.f(x))$ are r-isomorphisms of systems of groups, for all $s \geq 0$, $x \in X_s$, $k \geq 1$.

Observation: Suppose given a diagram of systems

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle \text{sect}\ \simeq}\downarrow & & \downarrow{\scriptstyle \simeq\ \text{sect}} \\ X_2 & \xrightarrow[f_2]{} & Y_2 \end{array}$$

Then f_1 is an r-equivalence iff f_2 is an r-equivalence.

Examples: Stability results. A sectionwise equivalence is a 0-equivalence.

A **controlled equivalence** is a map which is an r-equivalence for some $r \geq 0$.

Triangle axiom

Lemma 4.

Suppose given a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

If one of the maps is an r-equivalence, a second is an s-equivalence, then the third map is a $(r+s)$-equivalence.

Proof.

Set theory. □

Fibrations I

Lemma 5.

Suppose that $p : X \to Y$ is a sectionwise fibration of systems of Kan complexes and that p is an r-equivalence.
Then each lifting problem

$$\begin{array}{ccccc}
\partial\Delta^n & \xrightarrow{\alpha} & X_s & \xrightarrow{\sigma} & X_{s+2r} \\
\downarrow & & \downarrow & \overset{\theta}{\nearrow} & \downarrow p \\
\Delta^n & \xrightarrow[\beta]{} & Y_s & \xrightarrow[\sigma]{} & Y_{s+2r}
\end{array}$$

can be solved **up to shift** $2r$.

Fibrations II

Lemma 6.

Suppose that $p : X \to Y$ is a sectionwise fibration of systems of Kan complexes, and that all lifting problems

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & X_s & \xrightarrow{\sigma} & X_{s+r} \\ \downarrow & \theta & \downarrow & & \downarrow p \\ \Delta^n & \longrightarrow & Y_s & \xrightarrow[\sigma]{} & Y_{s+r} \end{array}$$

have solutions up to shift r. Then $p : X \to Y$ is an r-equivalence.

Proof.

If $p_*([\alpha]) = 0$ for $[\alpha] \in \pi_{n-1}(X_s, *)$, then there is a diagram on the left above. The existence of θ gives $\sigma_*([\alpha]) = 0$ in $\pi_{n-1}(X_{s+r}, *)$. □

Corollary 7.

Suppose given a pullback diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \longrightarrow & Y \end{array}$$

where p is a sectionwise fibration and an r-equivalence. Then the map p' is a sectionwise fibration and a $2r$-equivalence.

Cofibrations

Theorem 8.

Suppose that $i : A \to B$ is a sectionwise cofibration and an r-equivalence, and suppose given a pushout

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ B & \longrightarrow & D \end{array}$$

Then i_* is a sectionwise cofibration and a $2r$-equivalence.

Sketch (Whitehead theorem): There is a $2r$-interleaving

$$\begin{array}{ccc} A_s & \xrightarrow{\simeq} & FA_{s+2r} \\ \downarrow & \nearrow & \downarrow \\ B_s & \xrightarrow[\simeq]{} & FB_{s+2r} \end{array}$$

for a sectionwise fibrant model of i. The class of cofibrations admitting $2r$-interleavings is closed under pushout.

Category of cofibrations I

(A'): Suppose given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ & \searrow \quad \nearrow & \\ & B & \end{array}$$

If one of the maps is an r-equivalence, another is an s-equivalence, then the third is an $(r+s)$-equivalence.

(B): The composite of two cofibrations is a cofibration. Any isomorphism is a cofibration.

(C'): Cofibrations are closed under pushout. Given a pushout

$$\begin{array}{ccc} A & \to & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ B & \to & D \end{array}$$

with i a cofibration and r-equivalence, then i_* is a cofibration and a $2r$-equivalence.

Category of cofibrations II

(D): For any object A there is at least one cylinder object $A \otimes \Delta^1$.

(E): All objects are cofibrant.

This is an adjusted list of axioms for a category of cofibations structure — works for projective or sectionwise cofibrations.

There are standard formal (adjusted) outcomes:

Lemma 9 (left properness).

Suppose given a pushout

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[u_*]{} & D \end{array}$$

where i is a cofibration and u is an r-equivalence. Then u_* is a $2r$-equivalence.

There is also a **patching lemma**.

Example

Suppose given data sets $X \subset Y$, $X \subset W$ in a metric space Z such that $d_H(X, Y) < r$. Then $d_H(W, W \cup Y) < r$.

Here's a picture:

$$\begin{CD}
V_*(X) @>{2r}>> V_*(Y) \\
@VVV @VVV \\
V_*(W) @>>{2r}> V_*(W \cup Y)
\end{CD}$$

$V_*(W) \to V_*(W) \cup V_*(Y)$ is a $4r$-equivalence.

The map $V_*(W) \cup V_*(Y) \to V_*(W \cup Y)$ ("mapper" $\to$ "reality") is not an isomorphism, but it is a $6r$-equivalence.

- This is an excision statement for the Vietoris-Rips functor.
- The $6r$ bound is probably too coarse.

References

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