Persistent homotopy theory

Rick Jardine

University of Western Ontario

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Basic setup

X finite, $X \subset Z$, Z a metric space.

• $P_s(X)$ = poset of subsets $\sigma \subset X$ such that $d(x,y) \leq s$ for all $x,y \in \sigma$.

 $P_s(X)$ is the poset of non-degenerate simplices of the Vietoris-Rips complex $V_s(X)$. $BP_s(X)$ is **barycentric subdivision** of $V_s(X)$.

We have poset inclusions

$$\sigma: P_s(X) \subset P_t(X), \ s \leq t,$$

 $P_0(X) = X$, and $P_t(X) = \mathcal{P}(X)$ (all subsets of X) for t suff large.

• $k \ge 0$: $P_{s,k}(X) \subset P_s(X)$ subposet of simplices σ such that each element $x \in \sigma$ has at least k neighbours y such that $d(x,y) \le s$.

 $P_{s,k}(X)$ is the poset of non-degenerate simplices of the degree Rips complex $L_{s,k}(X)$, again the **barycentric subdivision**.

Stability results

Theorem 1 (Rips stability).

Suppose $X \subset Y$ in D(Z) such that $d_H(X,Y) < r$. There is a homotopy commutative diagram (homotopy interleaving)

$$\begin{array}{ccc} P_s(X) \stackrel{\sigma}{\rightarrow} P_{s+2r}(X) \\ i \downarrow & \theta & \forall i \\ P_s(Y) \stackrel{\rightarrow}{\rightarrow} P_{s+2r}(Y) \end{array}$$

Theorem 2.

Suppose $X \subset Y$ in D(Z) such that $d_H(X_{dis}^{k+1}, Y_{dis}^{k+1}) < r$. There is a homotopy commutative diagram

$$P_{s,k}(X) \stackrel{\sigma}{\Rightarrow} P_{s+2r,k}(X)$$

$$\downarrow i \qquad \qquad \downarrow i$$

$$P_{s,k}(Y) \stackrel{\sigma}{\Rightarrow} P_{s+2r,k}(Y)$$

Controlled equivalences

NB: $V_*(X) := BP_*(X)$ henceforth.

Suppose that $X \subset Y$ in D(Z) and we have a homotopy interleaving

$$V_{s}(X) \stackrel{\sigma}{\Rightarrow} V_{s+2r}(X)$$

$$i \downarrow \qquad \qquad \qquad \downarrow i$$

$$V_{s}(Y) \stackrel{\sigma}{\Rightarrow} V_{s+2r}(Y)$$

- 1) $i: \pi_0 V_*(X) \to \pi_0 V_*(Y)$ is a 2r-monomorphism: if i([x]) = i([y]) in $\pi_0 V_*(Y)$ then $\sigma[x] = \sigma[y]$ in $\pi_0 V_{s+2r}(X)$
- 2) $i: \pi_0 V_*(X) \to \pi_0 V_*(Y)$ is a 2r-epimorphism: given $[y] \in \pi_0 V_s(Y)$, $\sigma[y] = i[x]$ for some $[x] \in \pi_0 V_{s+2r}(X)$.
- 3) All $i: \pi_n(V_*(X), x) \to \pi_n(V_*(Y), i(x))$ are 2*r*-isomorphisms.

The map $i: V_*(X) \to V_*(Y)$ is a 2*r*-equivalence of systems.

Systems of spaces

A system of spaces is a functor $X:[0,\infty)\to s$ Set, aka. a diagram of simplicial sets with index category $[0,\infty)$.

A map of systems $X \to Y$ is a natural transformation of functors defined on $[0,\infty)$.

Examples

- 1) The functor $V_*(X)$, $s \mapsto V_s(X) = BP_s(X)$ is a system of spaces, for a data set $X \subset Z$.
- 2) If $X \subset Y \subset Z$ are data sets, the induced maps $P_s(X) \to P_s(Y)$, $V_s(X) \to V_s(Y)$ define maps of systems $P_*(X) \to P_*(Y)$ (posets) and $V_*(X) \to V_*(Y)$ (spaces).

Homotopy types

There are many ways to discuss homotopy types of systems. The oldest is the **projective structure** (Bousfield-Kan, 1972):

A map $f: X \to Y$ is a **weak equivalence** (resp. **fibration**) if each map $X_s \to Y_s$ is a weak equiv. (resp. fibration) of simplicial sets.

A map $A \to B$ is a **projective cofibration** if it has the left lifting property with respect all maps which are trivial fibrations.

Lemma 3.

Suppose that $X \subset Y \subset Z$ are data sets. Then $V_*(X) \to V_*(Y)$ is a projective cofibration.

The map $V_*(X) \to V_*(Y)$ is also a sectionwise cofibration, i.e. all maps $V_s(X) \to V_s(Y)$ are monomorphisms.

r-equivalences

Suppose that $f: X \to Y$ is a map of systems. Say that f is an r-equivalence if

- 1) the map $f:\pi_0(X)\to\pi_0(Y)$ is an r-isomorphism of systems of sets
- 2) the maps $f: \pi_k(X_s, x) \to \pi_k(Y_s, f(x))$ are *r*-isomorphisms of systems of groups, for all $s \ge 0$, $x \in X_s$, $k \ge 1$.

Observation: Suppose given a diagram of systems

$$X_1 \xrightarrow{f_1} Y_1$$

$$\text{sect } \simeq \psi \qquad \psi \simeq \text{ sect}$$

$$X_2 \xrightarrow{f_2} Y_2$$

Then f_1 is an r-equivalence iff f_2 is an r-equivalence.

Examples: Stability results. A sectionwise equivalence is a 0-equivalence.

A **controlled equivalence** is a map which is an r-equivalence for some r > 0.

Triangle axiom

Lemma 4.

Suppose given a commutative triangle

$$X \xrightarrow{f} Y$$

$$\downarrow g$$

$$Z$$

If one of the maps is an r-equivalence, a second is an s-equivalence, then the third map is a (r + s)-equivalence.

Proof.

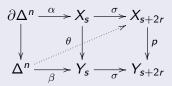
Set theory.

Fibrations I

Lemma 5.

Suppose that $p: X \to Y$ is a sectionwise fibration of systems of Kan complexes and that p is an r-equivalence.

Then each lifting problem



can be solved up to shift 2r.

Fibrations II

Lemma 6.

Suppose that $p: X \to Y$ is a sectionwise fibration of systems of Kan complexes, and that all lifting problems

have solutions up to shift r. Then $p: X \to Y$ is an r-equivalence.

Proof.

If $p_*([\alpha]) = 0$ for $[\alpha] \in \pi_{n-1}(X_s, *)$, then there is a diagram on the left above. The existence of θ gives $\sigma_*([\alpha]) = 0$ in $\pi_{n-1}(X_{s+r}, *)$.



Fibrations III

Corollary 7.

Suppose given a pullback diagram



where p is a sectionwise fibration and an r-equivalence. Then the map p' is a sectionwise fibration and a 2r-equivalence.

Cofibrations

Theorem 8.

Suppose that $i: A \rightarrow B$ is a sectionwise cofibration and an r-equivalence, and suppose given a pushout

$$\begin{array}{ccc} A \longrightarrow C & & \downarrow i_* \\ i \downarrow & & \downarrow i_* \\ B \longrightarrow D & \end{array}$$

Then i_* is a sectionwise cofibration and a 2r-equivalence.

Sketch (Whitehead theorem): There is a 2*r*-interleaving

$$A_{s} \xrightarrow{\simeq} FA_{s+2r}$$

$$\downarrow \qquad \qquad \downarrow$$

$$B_{s} \xrightarrow{\simeq} FB_{s+2r}$$

for a sectionwise fibrant model of i. The class of cofibrations admitting 2r-interleavings is closed under pushout.

Category of cofibrations I

(A'): Suppose given a commutative diagram



If one of the maps is an r-equivalence, another is an s-equivalence, then the third is an (r + s)-equivalence.

- **(B)**: The composite of two cofibrations is a cofibration. Any isomorphism is a cofibration.
- (C'): Cofibrations are closed under pushout. Given a pushout

$$\begin{array}{ccc}
A \longrightarrow C \\
\downarrow i_{*} & \downarrow i_{*} \\
B \longrightarrow D
\end{array}$$

with i a cofibration and r-equivalence, then i_* is a cofibration and a 2r-equivalence.

Category of cofibrations II

- **(D)**: For any object A there is at least one cylinder object $A \otimes \Delta^1$.
- (E): All objects are cofibrant.

This is an adjusted list of axioms for a category of cofibations structure — works for projective or sectionwise cofibrations.

There are standard formal (adjusted) outcomes:

Lemma 9 (left properness).

Suppose given a pushout

$$\begin{array}{ccc}
A \stackrel{u}{\Rightarrow} C \\
\downarrow \downarrow & \downarrow \\
B \xrightarrow[u_*]{} D
\end{array}$$

where i is a cofibration and u is an r-equivalence. Then u_* is a 2r-equivalence.

There is also a patching lemma.

Example

Suppose given data sets $X \subset Y$, $X \subset W$ in a metric space Z such that $d_H(X,Y) < r$. Then $d_H(W,W \cup Y) < r$.

Here's a picture:

$$V_{*}(X) \xrightarrow{2r} V_{*}(Y)$$

$$\downarrow \qquad \qquad \downarrow$$

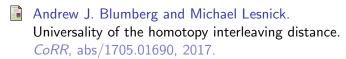
$$V_{*}(W) \xrightarrow{2r} V_{*}(W \cup Y)$$

 $V_*(W) \rightarrow V_*(W) \cup V_*(Y)$ is a 4*r*-equivalence.

The map $V_*(W) \cup V_*(Y) \to V_*(W \cup Y)$ ("mapper" \to "reality") is not an isomorphism, but it is a 6r-equivalence.

- This is an excision statement for the Vietoris-Rips functor.
- The 6r bound is probably too coarse.

References



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