

Clay Lecture

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Fields Institute

Persistent Homology  
From Chebyshev and Weierstrass  
to Gromov II.

Shmuel Weinberger

University of Chicago

Ideas at the interface of Pure and Applied Topology:

Persistent Homology

Metric Entropy

Navigation

Dimension reduction.

We will be applying these to problems of quantitative geometry, and the geometric understanding of function spaces. My hope is that there will be other applications of these in applied mathematics — although that part is only in the conceptual phase.

Connection to classical applied mathematics - which has a lot to do with nonlinear functions.

Lecture I review.

Studied the isoperimetric problem via Thom's method which

Associates a function  $F$  (with values in a strange space  $Th$ ) to each embedded manifold  $M$  in a sphere.

Finds a manifold  $W$  which bounds  $M$  from a null homotopy of  $F$ .

Leads to the questions:

1. Relating the complexity of  $M$  (e.g. Volume) to the map (which is the Lipschitz constant) — Quantitative embedding theory.

2. How complex must a function get during a homotopy?

3. One needs to “navigate” not just know a path exists. Some estimates come from entropy.

These aren't strong enough for us, but are useful for getting a baseline. We saw that entropy+smoothness gives a feeling for how many critical points are there with given depths.

Theorem (Chambers-Dotterer-Manin-W + Manin-W). If  $M$  is a closed oriented riemannian manifold with bounded curvature and injectivity radius, then, if  $M$  is a boundary,

$$\inf_{\partial W = M} \text{Vol}(W) \leq C \text{Vol}(M)^{\frac{n-1}{2}}$$

Items for emphasis.

$\text{PH}(\text{Function spaces})$  are interesting and beautiful (and perhaps useful)

Slogan: Bott and Morse versus Serre.

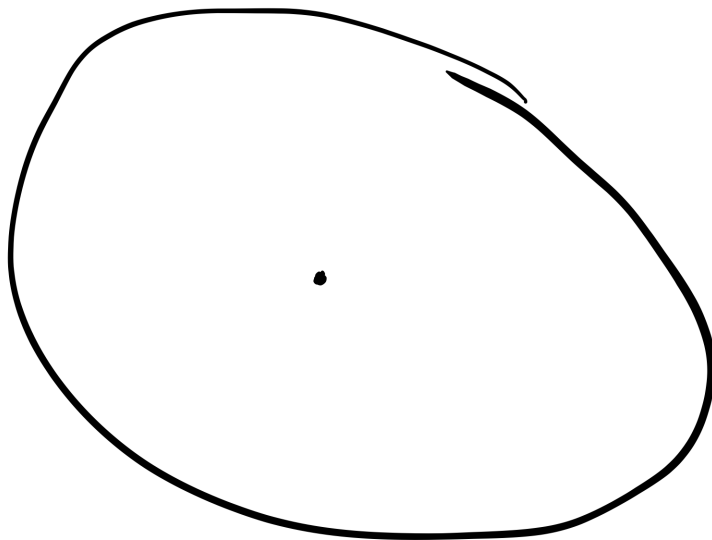
So we shall study  $\text{PH}(X, Y)$  in special cases.

We need  $X = S^N$  and  $Y = \text{Th}$ .

Let's first discuss  $Y = \mathbb{R}$ , then  $S^1$  (which isn't very different than non-positively curved spaces)

Then the case of  $X = S^1$ .

Finally, a bit of what we know about the more general situation.



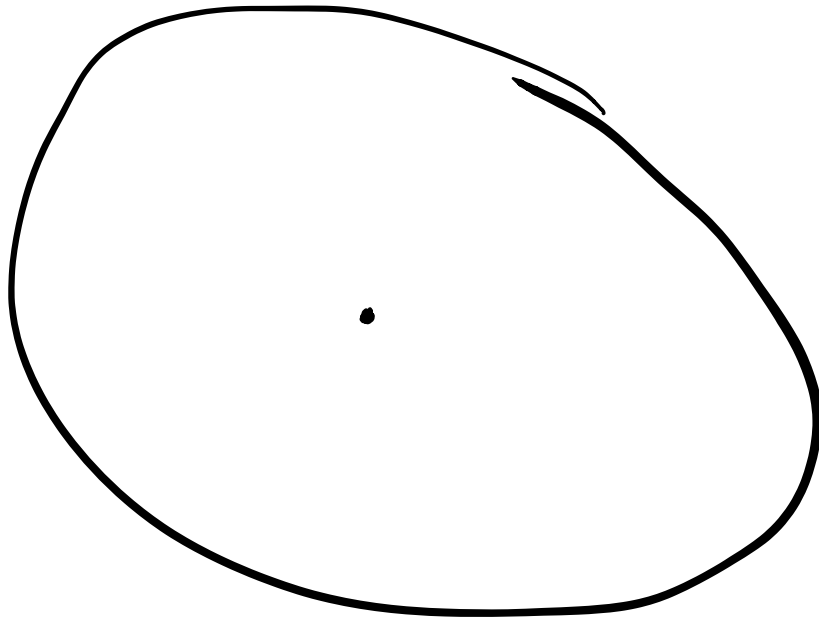
$$= \left\{ f \mid f: X \rightarrow [0,1] \right. \\ \left. \text{Lip}(f) \leq c \right\}$$

Navigate by

$$f_t = \left(\frac{t}{2}\right)f + \left(1 - \frac{t}{2}\right)g$$

Diameter = 2 .

PH has 1 infinite length bar.



$$= \left\{ f \mid f: X \rightarrow \mathbb{R} \text{ s.t. } f(0)=0 \right. \\ \left. \text{Lip}(f) \leq c \right\}$$

Navigate by

$$f_t = \left(\frac{t}{2c}\right)f + \left(1 - \frac{t}{2c}\right)g$$

Diameter =  $2c$ . (actually  $\cdot \text{Diam } X$  but never mind)

PH has 1 infinite length bar.

Maps  $[X : S']$  have "boring" PH

but — despite the compactness of  $S'$   
linear growing diameter.

Bars have bottoms, i.e. least Lipschitz constant  
for their realizations.

# bottoms with  $\text{Lip}(\cdot) \leq L \approx O(L^{b_1})$

(Nonpositively curved case: # is typically exponential  
and diameter is still linear)

What about

$$PH_0(\log L: \text{Map}(S^1: X) \rightarrow \mathbb{R}) \quad ?$$

Theorem: • This only depends on  $\pi, X$ .

- Having bars of uniformly bounded length

$\Leftrightarrow$  Any closed nullhomotopic geodesic can be contracted through curves that do not get more than a constant time larger.

- If  $\pi, X$  has unsolvable word problem (or superexponential Dehn function) there are arbitrarily long finite bars.

- So  $PH_0(\wedge X)$  is intimately tied to Geometric group theory
- Item (iii) is related to Entropy.
- One can get exponentially many such closed geodesics

with length  $< L$  by this method using stability and optimality if you want the geodesics to be local minima with a nontrivial basin.

- Suggests a connection between PH and other variational (AKA optimisation) problems. Indeed this is an important direction in Symplectic Topology and Hamiltonian dynamics. See the forthcoming book of Polterovich, Rosen, Samvelyan and Zhang. (AMS 2020)
- This theorem is a refinement of a variation of a result of Gromov.

Summary:

When the target is non-positively curved the PH is very simple: only infinite length bars.

The diameters are sometimes even bounded (but often linear)

(The counting is fun, algebra/geometry)

When the target is “computationally complex” the PH can be very complicated with very many very long bars.

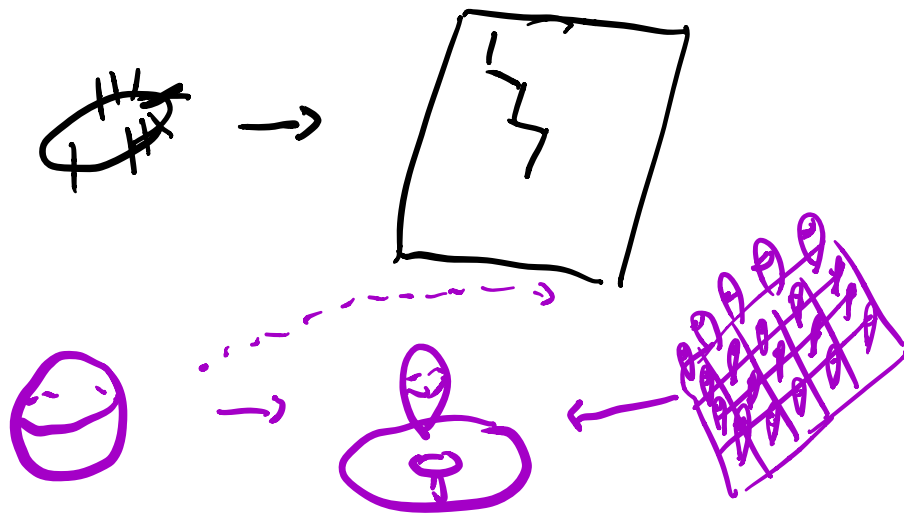
“Diameters” seem hard to get at.

## Observations

The Covering number of L-Lip Maps  $[X, Y]$

is  $\text{Exp}(L^{\dim X})$

Examples:



Theorem/Conjecture (Gromov, Manin-W). If  $X$  and  $Y$  are finite simplicial complexes and  $Y$  is simply connected (or has finite fundamental group) then the number of homotopy classes that have representatives with Lipschitz constant at most  $L$  is bounded by a polynomial in  $L$ .

(But, is not necessarily asymptotic to a polynomial)

Suggests that  $Y$  simply connected is understandable.

**Main Conclusion:** This seems to be true, but we're not there yet.

Theorem. (Ferry-W)

For a finite complex  $Y$  the following are equivalent:

There is a constant  $C(\dim X, Y)$  so that if  $f, g: X \rightarrow Y$  are homotopic  $L$ -Lipschitz maps then there is a  $CL$ -Lipschitz homotopy  $X \times [0, 1] \rightarrow Y$  between  $f$  and  $g$ .

All the homotopy groups of  $Y$  are finite. (If  $\pi_1 Y$  is finite, then it's algorithmic to tell).

Addendum (Block-W). In this case all the  $PH(\text{Log-Lip}(X:Y))$  are of bounded length.

The Ferry-W theorem together with some better technology is enough for a version of Gromov's isoperimetric inequality question when combined with careful embedding work.

Some of this is in the paper Chambers-Dotterer-Manin-W and the rest is in Manin-W and relies on work of Gromov-Guth.

For oriented manifolds this isn't enough.



Idea is to be guided some by Rational homotopy theory (Quillen, Sullivan).

Key papers are by CDMW JAMS, CMW GAFA, Manin G&T, Berdnikov, Manin-Berdnikov

Constructing nice embeddings.

- (1). Johnson-Lindenstrauss. (Clarkson)
- (2) Random projections or random embeddings.
- (3). Two stage random constructions (Gromov-Guth, Manin-W)

Potential directions of application.

- (a) navigating function spaces in applied math
- (b). Sobolev homology, and persistence. (Could be useful e.g. in liquid crystals, etc.)
- (c). Metastability phenomena. Evolutionary theory.
- (d). Open problems: Rather than just calculating the “phase space” a problem lives on — understand its PH. Develop methods other than logic and approximation. Make Serre more like Bott-Morse.

Search for and stable critical points - use entropy!

Don't dimension reduce as much as you can. Keep more local structure and be multi-scale.

