Clay Lecture
June 16, 2020
Fields Institute

Persistent Homology
From Chebyshev and Weierstrass
to Gromov.

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Ideas at the interface of Pure and Applied Topology:

Persistent Homology
Metric Entropy
Navigation
Dimension reduction.

We will be applying these to problems of quantitative geometry, and the geometric understanding of function spaces. My hope is that there will be other applications of these in applied mathematics — although that part is only in the conceptual phase.

Connection to classical applied mathematics - which has a lot to do with nonlinear functions.

I. Cobordism

(1). Thom's argument really fast.

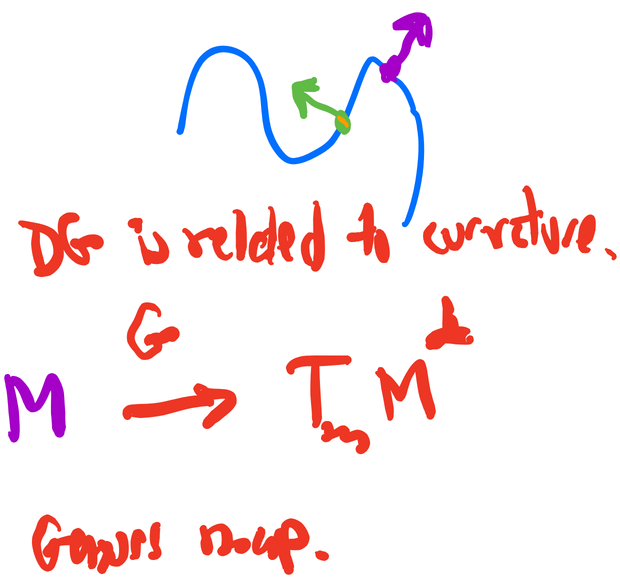
(Functions and geometries and the importance of learning functions which take value in strange spaces)

(2). Why do you care about PH(function spaces)?

(3). Why you need better: even a navigation result.

(4). Why you need dimension reduction.

(And the tension between dimension reduction and distortion.)



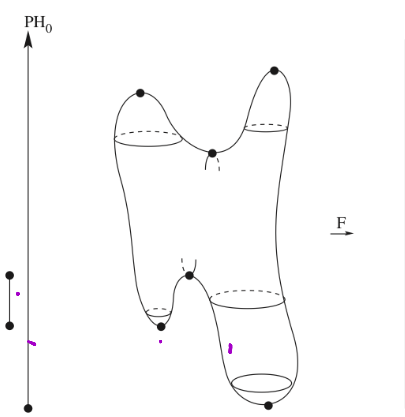
II. PH of functions.

(1). Continuous functions versus Lipschitz and Holder.

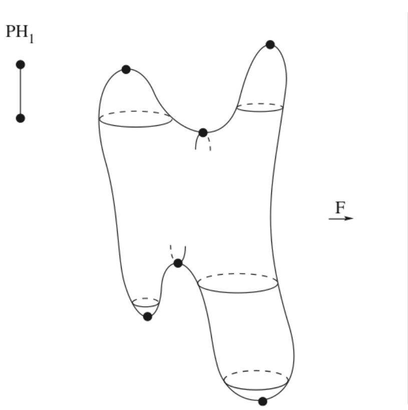
(2) Connection here between PH and entropy.

(3). Stability theorem (applications a la Chebyshev, Weierstrass)

Cohen-Steiner, Edelsbrunner, and Harer. FOCM.



This picture shows, on the left vertical axis, the birth of 0-dimensional homology classes (essentially components) and their deaths (the coalescing of components). For 1-dimensional homology it looks like this:



Features:

① Parametrization invariant.

② Enables the use of C^0 features to measure a C^2-notion

③ Stability properties.

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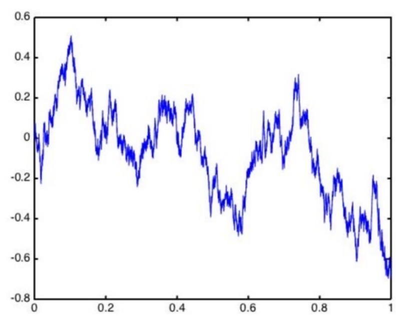


← Brownian Motion

Chebyshev Exercises.

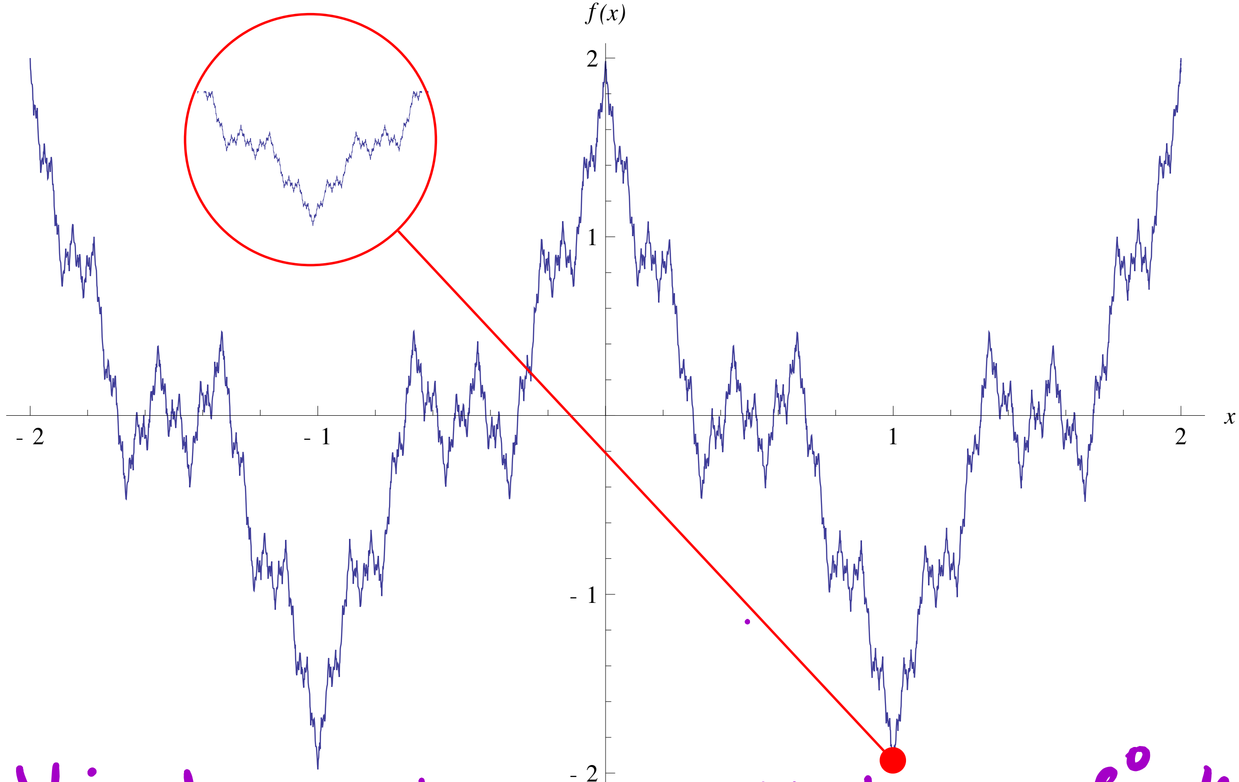
- How closely can you approximate x^n on $[-1, 1]$ by P_{n-1}? $\frac{1}{2^{n-1}}$

- What about $\cos(nx)$ by $T_{n-1}(S^1)$?

  L^2 vs L^∞.

  1 by orthonormality

  1 by distance between PH(f) and any T_{n-1} function.



Weierstrass and small bar stability versus C^0-large bar stability.

$$\sum 3^{-n} \sin n^n x$$ is nowhere Hölder.

Theorem (Baryshnikov - W) Holder function

The generic $C^{0,\alpha}(M^n; \mathbb{R})$ function has persistence bars with length

$$\sum l_i^{\frac{n}{\alpha}} \quad \text{converging}$$

and no smaller power

Based on Cohen-Steiner, Edelsbrunner, Harer, Mileyko FOCM 2010

The key point is the interplay between entropy of the underlying space, and the the modulus of continuity (= predictability) of the function.

$|f(x)-f(y)| \le C|x-y|^{\alpha}$ (can't be too rough)

$\sum \text{length}^{\frac{1}{\alpha}} \le C_0[0,1]$ max + min that

Next time we will

(1) Explain a bit about PH of function spaces
And how this connects to variational problems and entropy.

(2) What more needs to be done to prove isoperimetric inequalities.

And

(3) Try to formulate some lessons about how one can look for similar phenomena in other applications of these ideas.