Integral Gross-Stark conjecture and explicit formulae for Brumer-Stark units

(joint with Samit Dasgupta)

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Plan

1. Brumer's Conjecture.
2. Stark’s Conjecture.
3. Application to Hilbert’s 12th Problem.
The set up

\( F \) - totally real number field.
\( H \) - CM field, finite abelian extension of \( F \).
\( G = Gal(H/F) \).
\( R \) - all infinite places of \( F \) and all primes that ramify in \( H \).
\( q \) - a place of \( F \) of “large" residue characteristic (no non-trivial roots of unity in \( F \) congruent to 1 modulo \( q \)).
Brumer’s conjecture

Stickelberger element
(Deligne–Ribet, Cassou-Noguès) There exists $\Theta_{q}^{H/F} \in \mathbb{Z}[G]$ such that $\chi(\Theta_{q}^{H/F}) = L_{R}(\chi^{-1}, 0)(1 - \chi^{-1}(q)N_{q})$.

Class group
$Cl_{q}(H) = \frac{\text{Fractional ideal of } H \text{ prime to } q}{\text{Principal ideals } (\alpha) \text{ with } \alpha \equiv 1 \pmod{q}}$.

Brumer’s conjecture
$\Theta_{q}^{H/F}$ annihilates $Cl_{q}(H)$.

Strong Brumer’s conjecture
$\Theta_{q}^{H/F} \in \text{Fitt}_{\mathbb{Z}[G]}(Cl_{q}(H)^{\vee, -})$.

Remark: If $\chi$ is an even character, then $\chi(\Theta_{q}^{H/F}) = 0$. 
Stark’s conjecture

Let \( p \) be a finite place of \( F \) that splits completely in \( H \).

Let \( U = \{ x \in H^* : \mid x \mid_w = 1 \text{ for all } w \nmid p, \ u \equiv 1 \pmod{q} \} \).

Stark’s conjecture

Fix a prime \( \mathfrak{P} \) above \( p \) in \( H \). Then there exists \( u \in U \) such that

\[
\sum_{\sigma \in G} \left[ \sigma^{-1} \right] \text{ord}_{\mathfrak{P}}(\sigma(u)) = \Theta_{q}^{H/F}.
\]

If \( u \) exists it is unique.

There is a higher rank version of this conjecture due to Rubin (the Rubin–Stark conjecture) where one takes \( r \) primes in \( F \) that split completely in \( H \).
Brumer–Stark (Tate)

Brumer’s conjecture is equivalent to Stark’s conjecture.

**Brumer ⇔ Stark**

Let $p$ be a prime that splits completely in $H$. Fix a prime $\mathfrak{P}$ of $H$ above $p$. The class $[\mathfrak{P}]$ of $\mathfrak{P}$ in the class group $Cl_q(H)$ is annihilated by $\Theta_{q}^{H/F}$. Therefore $\Theta_{q}^{H/F}([\mathfrak{P}]) = (u)$ for the unit $u$ that we seek in Stark’s conjecture.

**Remark**

Stronger version with Fitting ideals implies the higher rank version i.e. Rubin–Stark conjecture.

**Theorem (Dasgupta-K)**

For all odd primes $p$

$$\Theta_{q}^{H/F} \in \text{Fitt}(Cl_q(H)^{\vee,-}).$$
Application to Hilbert’s 12th problem

Hilbert’s 12th problem

Explicit class field theory or Hilbert’s 12th problem asks for construction of $F^{ab}$ using information only in $F$. E.g. Kronecker–Weber theorem $\mathbb{Q}^{ab} = \sum_{n \in \mathbb{N}} \mathbb{Q}(e^{\frac{2\pi i}{n}})$.

Brumer–Stark units as generators

Fix an integral ideal $f$ of $F$. Let $p$ be a prime of $F$ not dividing $f$. Let $H_{f,p}$ be the maximal abelian CM extension of $F$ of conductor dividing $f$ and in which $p$ splits completely. Let $u_{f,p}$ be the Brumer–Stark unit in this setting. Then $u_{f,p}$ generates $H_{f,p}$.

It follows that $F^{ab} = F\left(\{u_{f,p}\}_{f,p}, \{\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{d-1}}\}\right)$, where $\alpha_i$ are a set of elements of $F$ whose signs in $\{\pm 1\}^d/(-1, \ldots, -1)$ are a $\mathbb{Z}/2\mathbb{Z}$-basis.

Brumer–Stark conjecture determines $u_{f,p}$ uniquely but does not tell anything about its construction i.e. as an explicit element in $F_p$ it is.
A conjecture of Dasgupta

Conjectures of Gross
The Gross–Stark conjecture and the tower of fields conjecture of Gross give more information about \( u_{f,p} \) but still does not determine it completely (as we will see later).

An explicit formula
A conjecture of Dasgupta and Spiess gives an explicit formula for Brumer-Stark units (and their higher rank analogues) in terms of Eisenstein cocycles. A special case: Let \( p = (p) \). Then

\[
\sigma(u) = p^{\zeta(\sigma,0)} \int_{O_p^*} xd\mu(\sigma)(x).
\]

Here \( \mu(\sigma) \) is an explicit measure constructed out of Eisenstein cocycles (or Shintani’s formulae in a special case) and \( \zeta(\sigma,0) \) is the value of a partial zeta function.
A conjecture for Dasgupta

Therefore, resolution of Dasgupta’s conjecture is a $p$-adic solution to Hilbert’s 12th problem for $F$.

Our way in to Dasgupta’s conjecture is through the tower of fields conjecture of Gross as we will see in the next talk.

The Gross–Stark conjecture is a special case of the tower of fields conjecture.
The Gross–Stark conjecture

$p$-adic $L$-function

For an odd character $\chi$ of $G$ there is a $p$-adic analytic function $L_p(\chi \omega, s)$ on $\mathbb{Z}_p \setminus \{1\}$ (where $\omega$ is the Teichmüller character) constructed by Cassou-Noguès and Delgine–Ribet satisfying

$$L_p(\chi \omega, n) = \prod_{p|\rho} (1 - \chi(p)Np^{-n})L(\chi \omega^n, n) \quad \text{for all } n \leq 0.$$ 

$L_p(\chi \omega, s)$ is also regular at $s = 1$ except possibly when $\chi \omega$ is trivial.

Trivial zeroes

It is trivial to see that if there is a $p \mid p$ such that $\chi(p) = 1$, then $L_p(\chi \omega, 0) = 0$.

Gross–Kuzmin conjecture

Let $r = \#\{p \mid p : \chi(p) = 1\}$, then $\operatorname{ord}_{s=0} L_p(\chi \omega, s) = r$. 
The Gross–Stark conjecture

Gross–Kuzmin conjecture
Let \( r = \#\{p \mid p : \chi(p) = 1\} \), then \( \text{ord}_{s=0} L_p(\chi \omega, s) = r \).

Theorem (Wiles \( p > 2 \), Dasgupta–Spiess)
\( \text{ord}_{s=0} L_p(\chi \omega, s) \geq r \).

Equality seems to be a problem in transcendence theory (similar to the Leoplodt conjecture). The Gross–Kuzmin conjecture is known by the Baker–Brumer theorem when \( r = 1 \).
The Gross–Stark conjecture

Let \( S = \{ p \mid p : \chi(p) = 1 \} = \{ p_1, \ldots, p_r \} \).

Put \( U = \left( O_H \left[ \frac{1}{p} \right] \right)^*( \otimes E)^{(\chi^{-1})} \), where \( E \) is a sufficiently large extension of \( \mathbb{Q}_p \). Then \( \dim_E U = r \) (Dirichlet’s unit theorem). Fix an \( E \)-basis \( \{ u_1, \ldots, u_r \} \) of \( U \). Fix a prime \( \mathfrak{P}_i \) of \( H \) above \( p_i \).

The Gross–Stark conjecture

We have

\[
\frac{L_p^{(r)}(\chi \omega, 0)}{r! L(\chi, 0) \prod p \mid p: \chi(p) \neq 1 (1 - \chi(p))} = \frac{\det (\log_p (N_{F_{p_i}/\mathbb{Q}_p}(u_j)))_{i,j}}{\det (\ord_{\mathfrak{P}_i}(u_j))_{i,j}}
\]

The special case of \( r = 1 \)

Put \( p = p_1 \) and \( u = u_1 \). The Gross–Stark conjecture gives a formula for \( N_{F_{p}/\mathbb{Q}_p}(u) \) in terms of the leading term of the \( p \)-adic \( L \)-function.
Gross-Stark conjecture

Theorem (Dasgupta–Darmon–Pollack, Ventullo, Dasgupta–K–Ventullo)

The Gross–Stark conjecture holds.

\[
\frac{L_p^{(r)}(\chi \omega, 0)}{r!L(\chi, 0) \prod_{p | \chi(p) \neq 1} (1 - \chi(p))} = \frac{\det(\log_p(N_{F_p}/\mathbb{Q}_p(u_j)))_{i,j}}{\det(\text{ord}_{\mathfrak{p}_i}(u_j))_{i,j}}.
\]

There is a reformulation of the Gross–Stark conjecture that motivates formulation of the tower of fields conjecture.

We end with this reformulation and in the next talk I will present the tower of fields conjecture, sketch how it implies Dasgupta’s conjecture on explicit formulae for Brumer–Stark units and sketch a proof of the Gross–Stark conjecture.
Reformulation of the Gross–Stark conjecture

Let $H_{cyc}$ be the cyclotomic $\mathbb{Z}_p$-extension of $H$. Put
\[ \Gamma = \text{Gal}(H_{cyc}/H) \cong \mathbb{Z}_p. \]
Then we have the tower of fields $H_{cyc}/H/F$. For every $p_i \in S$, there is
the Artin map
\[ \text{rec}_i : F_{p_i}^* \cong H_{p_i}^* \to \Gamma \subseteq \text{Gal}(H_{cyc}/F). \]

Recall that $\chi$ is an odd character of $G$. Let $O = \mathbb{Z}_p[\chi]$. We have an
isomorphism $O[[\Gamma]] \cong O[[T]]$ (after fixing a topological generator of $\Gamma$).
Further, let $I$ be the augmentation ideal
\[ I = \ker(O[[\Gamma]] \to O) \cong (T). \]
Reformulation of the Gross–Stark conjecture

Then

$$I^s / I^{s+1} \cong \Gamma \cong \mathbb{Z}_p$$

for any \( s \geq 1 \). (1)

Under (1) for \( s = 1 \) the image of \( \text{rec}_i(u) - 1 \in I \) in \( \mathbb{Z}_p \) is

\[ \log_p(N_{F_p/Q_p}(u)). \]

Consider the image \( \chi(\Theta_{q}^{H_{cyc}/F}) \in O[[\Gamma]] \) of \( \Theta_{q}^{H_{cyc}/F} \) under the map \( \chi : O[[\text{Gal}(H_{cyc}/F)]] \to O[[\Gamma]]. \)

By the result of Dasgupta–Spiess \( \chi(\Theta_{q}^{H_{cyc}/F}) \) belongs to \( I' \). Under the isomorphism (1) above the image of \( \chi(\Theta_{q}^{H_{cyc}/F}) \) is

\[ (1 - \chi(q)Nq)L_p^{(r)}(\chi \omega, 0)/r!. \]

The Gross–Stark conjecture \((r = 1)\)

Let \( u \) be the Brumer–Stark unit. Then

\[ \chi(\Theta_{q}^{H_{cyc}/F}) = \sum_{\sigma \in G} \chi(\sigma^{-1})(\text{rec}(\sigma(u)) - 1) \quad \text{in } I/I^2. \]
Thank you!