Integral Gross-Stark conjecture and explicit formulae for Brumer-Stark units

(joint with Samit Dasgupta)

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Plan

1. Brumer's Conjecture.
2. Stark's Conjecture.
3. Application to Hilbert's 12th Problem.
4. Gross-Stark conjecture.

The set up

F - totally real number field.
H - CM field, finite abelian extension of F.
$G = Gal(H/F)$.
R - all infinite places of F and all primes that ramify in H.
$\mathfrak{q}$ - a place of F of “large" residue characteristic (no non-trivial roots of unity in F congruent to 1 modulo $\mathfrak{q}$).

Brumer's conjecture

Stickelberger element

(Deligne–Ribet, Cassou-Noguès) There exists $\Theta_{\mathfrak{q}}^{H/F} \in \mathbb{Z}[G]$ such that $\chi(\Theta_{\mathfrak{q}}^{H/F}) = L_R(\chi^{-1}, 0)(1 - \chi^{-1}(\mathfrak{q})N\mathfrak{q})$.

Class group

$$Cl_{\mathfrak{q}}(H) = \frac{\text{Fractional ideal of } H \text{ prime to } \mathfrak{q}}{\text{Principal ideals } (\alpha) \text{ with } \alpha \equiv 1 \pmod{\mathfrak{q}}}.$$

Brumer's conjecture

$\Theta_{\mathfrak{q}}^{H/F}$ annihilates $Cl_{\mathfrak{q}}(H)$.

Strong Brumer's conjecture

$\Theta_{\mathfrak{q}}^{H/F} \in \mathrm{Fitt}_{\mathbb{Z}[G]}(Cl_{\mathfrak{q}}(H)^{\vee,-})$.

Remark: If χ is an even character, then $\chi(\Theta_{\mathfrak{q}}^{H/F}) = 0$.

Stark's conjecture

Let $\mathfrak{p}$ be a finite place of F that splits completely in H.

Let $U = \{x \in H^* : |x|_w = 1 \text{ for all } w \nmid \mathfrak{p}, u \equiv 1 \pmod{\mathfrak{q}}\}$.

Stark's conjecture

Fix a prime $\mathfrak{P}$ above $\mathfrak{p}$ in H. Then there exists $u \in U$ such that

$$\sum_{\sigma \in G} [\sigma^{-1}] \mathrm{ord}_{\mathfrak{P}}(\sigma(u)) = \Theta_{\mathfrak{q}}^{H/F}.$$

If u exists it is unique.
There is a higher rank version of this conjecture due to Rubin (the Rubin–Stark conjecture) where one takes r primes in F that split completely in H.

Brumer–Stark (Tate)

Brumer's conjecture is equivalent to Stark's conjecture.

Brumer $\implies$ Stark

Let $\mathfrak{p}$ be a prime that splits completely in H. Fix a prime $\mathfrak{P}$ of H above $\mathfrak{p}$. The class $[\mathfrak{P}]$ of $\mathfrak{P}$ in the class group $Cl_{\mathfrak{q}}(H)$ is annihilated by $\Theta_{\mathfrak{q}}^{H/F}$. Therefore $\Theta_{\mathfrak{q}}^{H/F}([\mathfrak{P}]) = (u)$ for the unit u that we seek in Stark's conjecture.

Remark

Stronger version with Fitting ideals implies the higher rank version i.e. Rubin–Stark conjecture.

Theorem (Dasgupta-K)

For all odd primes p

$$\Theta_{\mathfrak{q}}^{H/F} \in \mathrm{Fitt}(Cl_{\mathfrak{q}}(H)^{\vee,-}).$$

Application to Hilbert's 12th problem

Hilbert's 12th problem

Explicit class field theory or Hilbert's 12th problem asks for construction of F^{ab} using information only in F. e.g. Kronecker–Weber theorem $\mathbb{Q}^{ab} = \sum_{n\in\mathbb{N}} \mathbb{Q}(e^{\frac{2\pi i}{n}})$.

Brumer–Stark units as generators

Fix an integral ideal $\mathfrak{f}$ of F. Let $\mathfrak{p}$ be a prime of F not dividing $\mathfrak{f}$. Let $H_{\mathfrak{f},\mathfrak{p}}$ be the maximal abelian CM extension of F of conductor dividing $\mathfrak{f}$ and in which $\mathfrak{p}$ splits completely. Let $u_{\mathfrak{f},\mathfrak{p}}$ be the Brumer–Stark unit in this setting. Then $u_{\mathfrak{f},\mathfrak{p}}$ generates $H_{\mathfrak{f},\mathfrak{p}}$.

It follows that $F^{ab} = F\left(\{u_{\mathfrak{f},\mathfrak{p}}\}_{\mathfrak{f},\mathfrak{p}}, \{\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_{d-1}}\}\right)$, where α_i are a set of elements of F whose signs in $\{\pm 1\}^d/(-1,\ldots,-1)$ are a $\mathbb{Z}/2\mathbb{Z}$-basis.

Brumer–Stark conjecture determines $u_{\mathfrak{f},\mathfrak{p}}$ uniquely but does not tell anything about its construction i.e. as an explicit element in $F_{\mathfrak{p}}$ it is.

A conjecture of Dasgupta

Conjectures of Gross

The Gross–Stark conjecture and the tower of fields conjecture of Gross give more information about $u_{\mathfrak{f},\mathfrak{p}}$ but still does not determine it completely (as we will see later).

An explicit formula

A conjecture of Dasgupta and Spiess gives an explicit formula for Brumer-Stark units (and their higher rank analogues) in terms of Eisenstein cocycles. A special case: Let $\mathfrak{p} = (p)$. Then

$$\sigma(u) = p^{\zeta(\sigma,0)} \times\!\!\!\!\!\!\int_{O_p^*} x d\mu(\sigma)(x).$$

Here $\mu(\sigma)$ is an explicit measure constructed out of Eisenstein cocycles (or Shintani's formulae in a special case) and $\zeta(\sigma,0)$ is the value of a partial zeta function.

A conjecture for Dasgupta

Therefore, resolution of Dasgupta's conjecture is a p-adic solution to Hilbert's 12th problem for F.

Our way in to Dasgupta's conjecture is through the tower of fields conjecture of Gross as we will see in the next talk.

The Gross–Stark conjecture is a special case of the tower of fields conjecture.

The Gross–Stark conjecture

p-adic L-function

For an odd character χ of G there is a p-adic analytic function $L_p(\chi\omega, s)$ on $\mathbb{Z}_p \setminus \{1\}$ (where ω is the Teichmüller character) constructed by Cassou-Noguès and Delgine–Ribet satisfying

$$L_p(\chi\omega, n) = \prod_{\mathfrak{p}|p}(1 - \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-n})L(\chi\omega^n, n) \qquad \text{for all } n \leq 0.$$

$L_p(\chi\omega, s)$ is also regular at $s = 1$ except possibly when $\chi\omega$ is trivial.

Trivial zeroes

It is trivial to see that if there is a $\mathfrak{p} \mid p$ such that $\chi(\mathfrak{p}) = 1$, then $L_p(\chi\omega, 0) = 0$.

Gross–Kuzmin conjecture

Let $r = \#\{\mathfrak{p} \mid p : \chi(\mathfrak{p}) = 1\}$, then $\mathrm{ord}_{s=0} L_p(\chi\omega, s) = r$.

The Gross–Stark conjecture

Gross–Kuzmin conjecture

Let $r = \#\{\mathfrak{p} \mid p : \chi(\mathfrak{p}) = 1\}$, then $\mathrm{ord}_{s=0} L_p(\chi\omega, s) = r$.

Theorem (Wiles $p > 2$, Dasgupta–Spiess)

$\mathrm{ord}_{s=0} L_p(\chi\omega, s) \geq r$.

Equality seems to be a problem in transcendence theory (similar to the Leoplodt conjecture). The Gross–Kuzmin conjecture is known by the Baker–Brumer theorem when $r = 1$.

The Gross–Stark conjecture

Let $S = \{\mathfrak{p} \mid p : \chi(\mathfrak{p}) = 1\} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Put $U = \left(O_H\left[\frac{1}{p}\right]^* \otimes E\right)^{(\chi^{-1})}$, where E is a sufficiently large extension of $\mathbb{Q}_p$. Then $\dim_E U = r$ (Dirichlet's unit theorem). Fix an E-basis $\{u_1, \ldots, u_r\}$ of U. Fix a prime $\mathfrak{P}_i$ of H above $\mathfrak{p}_i$.

The Gross–Stark conjecture

We have

$$\frac{L_p^{(r)}(\chi\omega, 0)}{r!L(\chi, 0)\prod_{\mathfrak{p}|p:\chi(\mathfrak{p})\neq 1}(1-\chi(\mathfrak{p}))} = \frac{\det(\log_p(\mathrm{N}_{F_{\mathfrak{p}_i}/\mathbb{Q}_p}(u_j))_{i,j}}{\det(\mathrm{ord}_{\mathfrak{P}_i}(u_j))_{i,j}}$$

The special case of $r = 1$

Put $\mathfrak{p} = \mathfrak{p}_1$ and $u = u_1$. The Gross–Stark conjecture gives a formula for $\mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)$ in terms of the leading term of the p-adic L-function.

Gross-Stark conjecture

Theorem (Dasgupta–Darmon–Pollack, Ventullo, Dasgupta–K–Ventullo)

The Gross–Stark conjecture holds.

$$\frac{L_p^{(r)}(\chi\omega,0)}{r!L(\chi,0)\prod_{\mathfrak{p}|p:\chi(\mathfrak{p})\neq 1}(1-\chi(\mathfrak{p}))} = \frac{\det(\log_p(\mathrm{N}_{F_{\mathfrak{p}_i}/\mathbb{Q}_p}(u_j))_{i,j}}{\det(\mathrm{ord}_{\mathfrak{P}_i}(u_j))_{i,j}}.$$

There is a reformulation of the Gross–Stark conjecture that motivates formulation of the tower of fields conjecture.

We end with this reformulation and in the next talk I will present the tower of fields conjecture, sketch how it implies Dasgupta's conjecture on explicit formulae for Brumer–Stark units and sketch a proof of the Gross–Stark conjecture.

Reformulation of the Gross–Stark conjecture

Let H_{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of H. Put $\Gamma = \text{Gal}(H_{\text{cyc}}/H) \cong \mathbb{Z}_p$.
Then we have the tower of fields $H_{\mathrm{cyc}}/H/F$. For every $\mathfrak{p}_i \in S$, there is the Artin map

$$\mathrm{rec}_i : F_{\mathfrak{p}_i}^* \cong H_{\mathfrak{P}_i}^* \to \Gamma \subset \mathrm{Gal}(H_{\mathrm{cyc}}/F).$$

Recall that χ is an odd character of G. Let $O = \mathbb{Z}_p[\chi]$. We have an isomorphism $O[[\Gamma]] \cong O[[T]]$ (after fixing a topological generator of Γ). Further, let I be the augmentation ideal

$$I = ker(O[[\Gamma]] \to O) \cong (T).$$

Reformulation of the Gross–Stark conjecture

Then

$$I^s/I^{s+1} \cong \Gamma \cong \mathbb{Z}_p \quad \text{for any } s \geq 1. \tag{1}$$

Under (1) for $s = 1$ the image of $\mathrm{rec}_i(u) - 1 \in I$ in $\mathbb{Z}_p$ is $\log_p(\mathrm{N}_{F_{\mathfrak{p}_i}/\mathbb{Q}_p}(u))$.

Consider the image $\chi(\Theta_{\mathfrak{q}}^{H_{\mathrm{cyc}}/F}) \in O[[\Gamma]]$ of $\Theta_{\mathfrak{q}}^{H_{\mathrm{cyc}}/F}$ under the map $\chi : O[[\mathrm{Gal}(H_{\mathrm{cyc}}/F)]] \to O[[\Gamma]]$.

By the result of Dasgupta–Spiess $\chi(\Theta_{\mathfrak{q}}^{H_{\mathrm{cyc}}/F})$ belongs to I^r. Under the isomorphism (1) above the image of $\chi(\Theta_{\mathfrak{q}}^{H_{\mathrm{cyc}}/F})$ is $(1 - \chi(\mathfrak{q})N\mathfrak{q})L_p^{(r)}(\chi\omega, 0)/r!$.

The Gross–Stark conjecture ($r = 1$)

Let u be the Brumer–Stark unit. Then

$$\chi(\Theta_{\mathfrak{q}}^{H_{\mathrm{cyc}}/F}) = \sum_{\sigma \in G} \chi(\sigma^{-1})(\mathrm{rec}(\sigma(u)) - 1) \qquad \text{in } I/I^2.$$

Thank you!