# ON THE SUBRING OF SPECIAL CYCLES

STEPHEN S. KUDLA

Fields Number Theory Seminar
June 17, 2020

## 1. Orthogonal Shimura varieties

I am going to report on some results about the structure of the ring generated by the cohomology classes of special cycles in orthogonal Shimura varieties over totally real fields. These results are from the second of a pair of recent preprints:

*Remarks on generating series for special cycles*, arXiv:1908.08390v1.

*On the subring of special cycles*, arXiv:2001.09068v1.

**The Shimura variety:**

$$F = \text{totally real field}, \ |F:\mathbb{Q}| = d$$

$$V = \text{quadratic space over } F,$$

$$\text{sig}(V) = (m,2)^{d_+} \times (m+2,0)^{d-d_+},$$

$$\Sigma(V)_+ = \{\, \sigma \mid \text{sig}(V_\sigma) = (m,2),\ V_\sigma = V \otimes_{F,\sigma} \mathbb{R} \,\}$$

$$G = R_{F/\mathbb{Q}}\text{GSpin}(V)$$

$$D_\sigma = \{\, z \in \text{Gr}_2^o(V_\sigma) \mid \ Q\,|_z < 0 \,\}, \qquad \dim D_\sigma = m$$

$$D = \prod_{\sigma \in \Sigma_+} D_\sigma, \qquad \dim D = md_+,\ D^+ = \text{one component}$$

$$S_K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K, \qquad K \subset G(\mathbb{A}_f), \text{compact open}$$

**Components:**

$$G(\mathbb{A}_f) = \coprod_j G(\mathbb{Q})_+ g_j K$$

$$S_K \simeq \coprod_j \Gamma_j \backslash D, \qquad \Gamma_j = G(\mathbb{Q})_+ \cap g_j K g_j^{-1}$$

**Assume:** $V$ anisotropic, $d_+ \geq 1$, $K$ neat

$$S_K = \text{smooth projective variety over } \mathbb{C}.$$

**Cohomology:**

$$H^\bullet(S_K) = \bigoplus_{r=0}^{2md_+} H^r(S_K, \mathbb{C})$$

$$\deg_K : H^{2md_+}(S_K) \longrightarrow \mathbb{C}$$

It is a **graded ring under cup product** with an **inner product**

$$\langle z, z' \rangle = \deg_K(z \cup z') = \mathrm{vol}(K/K \cap Z(\mathbb{Q}))\ \deg_K^\natural(z \cup z').$$

Passing to the limit, for the covers $K' \subset K$, $\mathrm{pr} : S_{K'} \longrightarrow S_K$,

$$H^\bullet(S) = \varinjlim_K H^\bullet(S_K), \qquad H^r(S_K) = H^r(S)^K.$$

**Components:** $G = R_{F/\mathbb{Q}}\mathrm{GSpin}(V)$

$$\pi_0(S_K) \simeq G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K \simeq F^\times_{\mathbb{A}_f}/F^\times_+ \nu(K),$$

$$\nu : G \to R_{F/\mathbb{Q}}\mathbb{G}_m, \quad \text{the spinor norm}$$

$$F^\times_+ = \text{the group of totally positive elements in } F^\times.$$

$$H^0(S) = C(F^\times_{\mathbb{A}_f}/F^\times_+, \mathbb{C}) = \varinjlim_K C(F^\times_{\mathbb{A}_f}/F^\times_+\nu(K), \mathbb{C})$$

$$= \text{the space of continuous complex valued functions on } F^\times_{\mathbb{A}_f}/F^\times_+$$

so there are classes

$$1\!\!1 \in H^0(S), \qquad \boldsymbol{\chi} \in H^0(S).$$

**Co-tautological bundle:** The following variant of the Kähler class will play a role:

$$D_\sigma \simeq \{\ w \in (V_\sigma)_\mathbb{C} \mid (w,w) = 0,\ (w, \bar{w}) < 0\ \}/\mathbb{C}^\times \quad \subset\ \mathbb{P}((V_\sigma)_\mathbb{C})$$

$$\mathcal{L}_\sigma = \mathrm{pr}_\sigma^*(O(-1)|_{D_\sigma}) = \text{a tautological bundle on } D$$

$$\mathrm{pr}_\sigma : D \longrightarrow D_\sigma$$

$\mathcal{L}_\sigma$ descends to $S_K$ and $S$

$$\boldsymbol{c}_S = \prod_\sigma c_1(\mathcal{L}_\sigma^\vee) \quad \in H^{2d_+}(S), \qquad \text{product Chern class.}$$

## 2. Weighted special cycles

**For a subspace:**

$$\begin{aligned}
W &= \text{totally positive definite } F\text{-subspace of } V\\
D^+_W &= \prod_\sigma D^+_{\sigma,W}\\
D^+_{\sigma,W} &= \{ z\in D^+_\sigma \mid z\subset W^\perp\otimes_{F,\sigma}\mathbb{R}\ \}\\
\operatorname{codim} D^+_W &= r(W)\,d_+, \qquad r(W)=\dim_F W,\\
Z(W)_\Gamma &= \pi_\Gamma(D^+_W), \qquad \text{an algebraic cycle of codimension } nd_+.\\
\pi_\Gamma &: D^+ \longrightarrow \Gamma\backslash D^+
\end{aligned}$$

**For a frame:** $x=[x_1,\dots,x_n]\in V(F)^n$**,** $W(x)=\operatorname{span}(x_1,\dots,x_n)$**,**

$$[Z(x)]_\Gamma = \begin{cases}[Z(W(x))]_\Gamma\cdot \boldsymbol{c}_\Gamma^{n-r(x)} & \text{if } W(x) \text{ is totally positive definite,}\\ 0 & \text{otherwise.}\end{cases}$$

$$[Z(x)]_\Gamma\in H^{2nd_+}(\Gamma\backslash D^+), \qquad \text{Hodge type } (nd_+,nd_+).$$

**Weighted special cycles:**
For

$$\begin{aligned}
&T\in \operatorname{Sym}_n(F)_{\geq 0}\\
&\varphi\in S(V(\mathbb{A}_f)^n)\\
Z(T,\varphi,K) := &\sum_j \sum_{\substack{x\in V(F)^n\\ Q(x)=T\\ \bmod \Gamma_j}} \varphi(g_j^{-1}x)\,[\,Z(x)]_{\Gamma_j} \qquad \in H^{2nd_+}(S_K)
\end{aligned}$$

Here recall that

$$S_K\simeq \coprod_j \Gamma_j\backslash D, \qquad \Gamma_j = G(\mathbb{Q})_+\cap g_jKg_j^{-1}.$$

**Example:**

$$Z(0,\varphi,K)=\boldsymbol{c}_S^n\cdot\varphi(0).$$

**Pullbacks:** This compatibility is a crucial property of the weighted cycles:

$$\begin{aligned}
\operatorname{pr}: S_{K'} &\longrightarrow S_K \qquad K'\subset K,\\
\operatorname{pr}^*(Z(T,\varphi,K)) &= Z(T,\varphi,K')\\
Z(T,\varphi) &\in H^{2nd_+}(S).
\end{aligned}$$

▶ There is a very explicit **product formula** of weighted special cycles:

**Theorem A:** For $T_i \in \mathrm{Sym}_{n_i}(F)$ and $\varphi_i \in S(V(\mathbb{A}_f)^{n_i})$,

$$Z(T_1,\varphi_1)\cdot Z(T_2,\varphi_2) = \sum_{\substack{T\in \mathrm{Sym}_{n_1+n_2}(F)_{\geq 0}\\ T=\begin{pmatrix} T_1 & * \\ {}^t* & T_2\end{pmatrix}}} Z(T,\varphi_1\otimes\varphi_2). \tag{2.1}$$

▶ Thus, the $Z(T,\varphi)$'s together with the class $1\!\!1$ for $n=0$, defines **the subring of special cycles**:

$$\mathrm{SC}^\bullet(V)^\natural \subset H^{2\bullet d_+}(\mathrm{Sh}(V)).$$

Note the shift in degree.

▶ Since our access to the structure of these rings will be via intersection numbers, we introduce 'reduced' or 'numerical' versions.

Consider the restriction of the intersection pairing

$$\langle z, z'\rangle = \deg_K(z\cdot z')$$

to subring of special cycles.
By associativity of the cup product, the radical of this pairing on $\mathrm{SC}^\bullet(V)^\natural$ is an ideal.

**Definition.** The **reduced ring of special cycles** is the subquotient of the cohomology ring

$$\mathrm{SC}^\bullet(V) := \mathrm{SC}^\bullet(V)^\natural/\mathrm{Rad}.$$

The form $\langle\, ,\,\rangle$ then defines a non-degenerate pairing on $\mathrm{SC}^\bullet(V)$.

**Examples:**

$$z(T;\varphi) = z_V(T;\varphi) = \text{ the image of } Z(T;\varphi) \text{ in } \mathrm{SC}^\bullet(V),$$

$$\mathrm{SC}^0(V)^\natural = \mathbb{C}\,1\!\!1, \qquad \text{the degree 0 part, by definition.}$$

$$\mathrm{SC}^m(V)^\natural \cap \mathrm{Rad} = \ker(\deg : \mathrm{SC}^m(V)^\natural \to \mathbb{C}),$$

$$\mathrm{SC}^m(V) = \mathbb{C}\,1\!\!1^\vee, \qquad \langle\, 1\!\!1, 1\!\!1^\vee\,\rangle = 1.$$

For $T\in \mathrm{Sym}_m(F)$ and $\varphi\in S(V(\mathbb{A}_f)^m)$,

$$z(T,\varphi) = \deg(Z(T,\varphi))\cdot 1\!\!1^\vee.$$

▶ Recall that quadratic spaces over $F$ are determined by

$$\begin{aligned} &\dim(V) \\ &\det(V) \in F^{\times}/F^{\times,2} \\ \chi_V(x) &= (x, (-1)^{\frac{1}{2}(m+1)(m+2)} \det(V))_F \\ \mathrm{sig}(V_\sigma) &= \text{signature of } V_\sigma = V \otimes_{F,\sigma} \mathbb{R} \\ \epsilon_v(V) &= \pm 1 = \text{Hasse invariant} \\ 1 &= \prod_v \epsilon_v(V). \qquad (\star) \end{aligned}$$

At an archimedean place,

$$\epsilon_\sigma(V) = \begin{cases} -1 & \text{if } \mathrm{sig}(V_\sigma) = (m,2), \\ \ \ 1 & \text{if } \mathrm{sig}(V_\sigma) = (m+2,0). \end{cases} \qquad (\star\star)$$

In particular, if the finite part $V_f = V(\mathbb{A}_f)$ is fixed, then the signatures $(m,2)$ and $(m+2,0)$ can be chosen arbitrarily subject to the parity condition $(\star)$.

▶ Here is a surprising **comparison result**:

**Theorem B.** Suppose that $V$ and $V'$ are quadratic spaces over $F$ as above with $\chi_V = \chi_{V'}$ and

$$V \otimes_F \mathbb{A}_f \simeq V' \otimes_F \mathbb{A}_f.$$

By $(\star)$ and $(\star\star)$, this implies $d_+(V)$ and $d_+(V')$ have the same parity.
Then there is an isometry of graded rings

$$\mathrm{SC}^\bullet(V) \xrightarrow{\sim} \mathrm{SC}^\bullet(V'), \qquad z_V(T,\varphi) \xrightarrow{\sim} z_{V'}(T,\varphi'),$$

where

$$S(V(\mathbb{A}_f)^n) \xrightarrow{\sim} S(V'(\mathbb{A}_f)^n), \qquad \varphi \mapsto \varphi'.$$

▶ Note that

$$\dim \mathrm{Sh}(V) = m d_+(V)$$
$$\dim \mathrm{Sh}(V') = m d_+(V').$$

so the isomorphism of the theorem can involve a shift in dimensions.

▶ I do not know of a ‘geometric’ reason for such an isomorphism.

## 3. Dual forms and generating series

▶ There is a generating series for cohomology classes of special cycle classes:

$$\phi_n(\tau;\varphi) = \sum_{T\in \mathrm{Sym}_n(F)_{\ge 0}} Z(T,\varphi)\cdot \boldsymbol{q}^T \quad \in \ H^{2nd_+}(S)[[\boldsymbol{q}]]$$

where

$$\tau = (\tau_\sigma)_{\sigma\in\Sigma} \in \mathfrak{H}_n^d, \qquad T\in \mathrm{Sym}_n(F), \qquad \Sigma = \mathrm{Hom}(F,\mathbb{R}),$$
$$\boldsymbol{q}^T = e(\sum_\sigma \mathrm{tr}(\sigma(T)\tau_\sigma)\,), \qquad e(t)=e^{2\pi i t}, \qquad \varphi\in S(V(\mathbb{A}_f)^n).$$

From my old work with Millson,

**Theorem:** (KM)

$$\phi_n(\tau;\varphi) = \text{the } \boldsymbol{q}\text{-expansion of a Hilbert-Siegel modular form}$$
$$\text{of parallel weight } (\frac{m}{2}+1,\dots,\frac{m}{2}+1).$$

▶ The crucial point behind this result is that the generating series is the image in cohomology of a theta series valued in the deRham complex.

I will be slightly 'impressionistic' in my description of this.

To construction 'geometric' theta functions, Millson and I wrote down
a Schwartz function valued in differential forms.

In the present situation:
▶ The **Schwartz form is**

$$\boldsymbol{\varphi}_\infty^{(n)} \ \in\ \big[\,S(V_\mathbb{R}^n)\otimes A^{(nd_+,nd_+)}(D)\,\big]^{G(\mathbb{R})},$$
$$= \bigotimes_{\sigma\in\Sigma_+(V)} \varphi_\sigma^{(n)} \ \otimes \bigotimes_{\sigma\notin\Sigma_+(V)} \varphi_{\sigma,+}^0$$

where, for $\sigma\in\Sigma_+(V)$,

$$\varphi_\sigma^{(n)} \ \in S(V_\sigma^n)\otimes A^{(n,n)}(D_\sigma)$$

is the Schwartz form for $V_\sigma$, and, for $\sigma\notin\Sigma_+(V)$, $\varphi_{\sigma,+}^0\in S(V_\sigma^n)$ is the Gaussian for $V_\sigma$.

These forms satisfy

$$g^*\varphi_\sigma^{(n)}(x) = \varphi(g^{-1}x), \qquad g\in \mathrm{SO}(V_\sigma).$$

**The Theta form:**

▶ Modularity of the generating series for the cohomology classes of special cycles can be proved using 'geometric' theta series.

For $\varphi \in S(V(\mathbb{A}_f)^n)^K$,

$$\boldsymbol{\varphi}_\infty^{(n)} \otimes \varphi \quad \in S(V(\mathbb{A})^n) \otimes A^{(n,n)}(D),$$

and the theta form

$$\theta(g';\varphi) = \sum_{x \in V(F)^n} \omega(g')(\boldsymbol{\varphi}_\infty^{(n)} \otimes \varphi)(x), \qquad g' \in \widetilde{G'}(\mathbb{A}) = \text{metaplectic group},$$

a closed $(nd_+, nd_+)$-form on $S_K$ and, as a function of $g'$, is left invariant for the (canonical) image of $G'(\mathbb{Q})$ in $\widetilde{G'}(\mathbb{A})$.

**Theorem:** (KM) The cohomology class of the theta form is the generating series

$$N(\det(v))^{-\frac{m+2}{4}} \left[ \theta(g'_\tau;\varphi) \right] = \phi_n(\tau;\varphi) = \sum_{T \in \mathrm{Sym}_n(F)_{\geq 0}} Z(T,\varphi)\, \boldsymbol{q}^T.$$

Here $g'_\tau = [g_\tau, 1]$ is an element of the metaplectic cover $\widetilde{G'}(\mathbb{R})$ where $g_\tau \in G'(\mathbb{R})$ has components

$$(g_\tau)_\sigma = \begin{pmatrix} 1 & u_\sigma \\ & 1 \end{pmatrix} \begin{pmatrix} a_\sigma & \\ & {}^t a_\sigma^{-1} \end{pmatrix}, \qquad g_\sigma(i) = \tau_\sigma = u_\sigma + i v_\sigma \in \mathfrak{H}_n, \quad v_\sigma = a_\sigma\, {}^t a_\sigma.$$

**Computing degrees:**

▶ This identity allows us to do computations with the special cycle cohomology classes.

The key is to use the **Siegel-Weil formula** to relate
degree integrals of theta forms to special values of Eisenstein series.

**Crucial Fact:** The Schwartz forms are compatible with wedge products so that, for $n = n_1 + n_2$,

$$\boldsymbol{\varphi}_\infty^{(n)} = \boldsymbol{\varphi}_\infty^{(n_1)} \wedge \boldsymbol{\varphi}_\infty^{(n_2)}.$$

Thus, $\tau_1 \in \mathfrak{H}_{n_1}^d$, $\tau_2 \in \mathfrak{H}_{n_2}^d$, and $\varphi_i \in S(V(\mathbb{A}_f)^{n_i})$, $i = 1$, 2,

$$\theta(g'_{\tau_1};\varphi_1) \wedge \theta(g'_{\tau_2};\varphi_2) = \theta(g'_\tau;\varphi_1 \otimes \varphi_2), \qquad \tau = \begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}.$$

▶ For $n_1+n_2=m$, we compute the inner product of the generating series:

$$
\begin{aligned}
&\langle\, \phi_{n_1}(\tau_1,\varphi_1), \phi_{n_2}(\tau_2,\varphi_2)\,\rangle \\
&\qquad = \deg(\phi_{n_1}(\tau_1,\varphi_1)\cup \phi_{n_2}(\tau_2,\varphi_2)) \\
&\qquad = \mathrm{vol}(K/K\cap Z(\mathbb{Q}))\int_{S_K}\theta(g'_{\tau_1};\varphi_1)\wedge\theta(g'_{\tau_2};\varphi_2) \\
&\qquad = (-1)^{md_+}N(\det(v_1)\det(v_2))^{-\frac{m+2}{4}}\int_{G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})}\theta(g'_\tau,g;\breve{\boldsymbol{\varphi}}_\infty^{(m)}\otimes\varphi_1\otimes\varphi_2)\,dg.
\end{aligned}
$$

Here in the last line, we are shifting from the integral of the top degree geometric theta series as a top degree differential form to the group integral with respect to Tamagawa measure.

▶ By the Siegel-Weil formula, such an integral is given by a special value of an Eisenstein series.

**Review of the Siegel-Weil formula**

Assume $m$ is even to avoid the extra notation for the metaplectic group, etc.

$$
\begin{aligned}
G' &= \mathrm{Sp}(n)/F \\
P' &= \text{Siegel parabolic, } n(b),\ m(a), \text{ etc.} \\
I_n(s,\chi_V) &= \text{degenerate principal series,} \\
\lambda_V: S(V(\mathbb{A})^n) &\longrightarrow I_n(s_0,\chi), \qquad \boldsymbol{\varphi}\mapsto\lambda_V(\boldsymbol{\varphi})(g')=\omega(g')\boldsymbol{\varphi}(0), \qquad s_0=\frac{1}{2}\dim V-\rho_n \\
\Phi(g',s;\boldsymbol{\varphi}) &= \omega(g')\boldsymbol{\varphi}(0)\cdot|a(g')|^{s-s_0} \\
&= \text{standard section attached to } \varphi, \\
E(g',s,\lambda_V(\boldsymbol{\varphi})) &= \sum_{\gamma\in P'(F)\backslash G'(F)}\Phi(\gamma g',s;\boldsymbol{\varphi}) \\
&= \text{the Eisenstein series, } \mathrm{Re}(s)>\rho_n=\frac{1}{2}(n+1).
\end{aligned}
$$

**Theorem:** (Siegel-Weil, KR 1988) Since $V$ is anisotropic, the Eisenstein series is holomorphic at $s=s_0$ and

$$E(g',s_0,\lambda_V(\boldsymbol{\varphi}))=\int_{O(V)(F)\backslash O(V)(\mathbb{A}_F)}\theta(g',g;\boldsymbol{\varphi})\,dg,$$

for $\mathrm{vol}(O(V)(F)\backslash O(V)(\mathbb{A}_F),dg)=1$.

**In fact:**

$$\int_{O(V)(F)\backslash O(V)(\mathbb{A}_F)}\theta(g',g;\boldsymbol{\varphi})\,dg=\frac{1}{2}\int_{\mathrm{SO}(V)(F)\backslash SO(V)(\mathbb{A}_F)}\theta(g',g;\boldsymbol{\varphi})\,d^Tg,$$

where $d^Tg$ is the Tamagawa measure on $\mathrm{SO}(V)(\mathbb{A}_F)$.

## 4. Inner products and products

**Computing inner products and products.**

► We clean up the notation with the definition of
the Hilbert-Siegel Eisenstein series of genus $m$ and weight $\frac{1}{2}m+1$:

$$E(\tau, s_0, \lambda_{V_f}(\varphi)) := (-1)^{md_+} 2\, N(\det(v))^{-\kappa/2} \cdot E(g'_\tau, s_0, \lambda_V(\breve{\boldsymbol{\varphi}}_\infty \otimes \varphi)).$$

for the top degree, where $\varphi \in S(V^m(\mathbb{A}_f))$.

**Notes:**
(1) The special value is now taken at the point $s_0 = \kappa - \rho_m = \frac{1}{2}$,
very close to the center of the critical strip!
(2) This is a holomorphic Siegel modular form of low weight!
The lowest holomorphic discrete series of genus $m$ has weight $m+1$.
For $m=1$, the weight is $\frac{3}{2}$, for $m=2$ weight 2, for $m=3$, weight $\frac{5}{2}$, etc.
(3) The archimedean component of this automorphic representation occurring here is an interesting lowest weight representation. It sits in the degenerate principal series.

► Combining the steps, we get
**a basic formula for the inner product of the generating series:**

$$\begin{aligned}&\langle\ \phi_{n_1}(\tau_1, \varphi_1), \phi_{n_2}(\tau_2, \varphi_2)\ \rangle \\ &\qquad = (-1)^{md_+} N(\det(v_1)\det(v_2))^{-\frac{m+2}{4}} \int_{G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})} \theta(g'_\tau, g; \breve{\boldsymbol{\varphi}}^{(m)}_\infty \otimes \varphi_1 \otimes \varphi_2)\, dg \\ &\qquad = E(\begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix}, \frac{1}{2}, \lambda_{V_f}(\varphi_1 \otimes \varphi_2)).\end{aligned}$$

It is a geometric version of the Rallis inner product formula.

Here we are using the fact that, for $n_1 + n_2 = m$,

$$\boldsymbol{\varphi}^{(n_1)}_\infty \wedge \boldsymbol{\varphi}^{(n_2)}_\infty = \boldsymbol{\varphi}^{(m)}_\infty = \breve{\boldsymbol{\varphi}}^{(m)}_\infty \boldsymbol{\Omega}^m.$$

for a scalar valued Schwartz function $\breve{\boldsymbol{\varphi}}^{(m)}_\infty \in \mathcal{S}(V^m_\sigma)$.

▶ Starting with the formula

$$\langle\, \phi_{n_1}(\tau_1, \varphi_1), \phi_{n_2}(\tau_2, \varphi_2)\,\rangle = E(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, \frac{1}{2}, \lambda_{V_f}(\varphi_1 \otimes \varphi_2)),$$

we obtain identities among the Fourier expansions of the two sides:

$$E(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, \frac{1}{2}, \lambda_{V_f}(\varphi_1 \otimes \varphi_2)) := \sum_{T_1 \in \mathrm{Sym}_{n_1}(F)_{\geq 0}} \sum_{T_2 \in \mathrm{Sym}_{n_2}(F)_{\geq 0}} A(T_1, T_2; \lambda_{V_f}(\varphi_1 \otimes \varphi_2))\, \boldsymbol{q}_1^{T_1}\, \boldsymbol{q}_2^{T_2}$$

and

$$\langle\, \phi_{n_1}(\tau_1, \varphi_1), \phi_{n_2}(\tau_2, \varphi_2)\,\rangle = \sum_{T_1 \in \mathrm{Sym}_{n_1}(F)_{\geq 0}} \sum_{T_2 \in \mathrm{Sym}_{n_2}(F)_{\geq 0}} \langle\, z(T_1, \varphi_1), z(T_2, \varphi_2)\,\rangle\, \boldsymbol{q}_1^{T_1}\, \boldsymbol{q}_2^{T_2}.$$

**Theorem C:** For $n_1 + n_2 = m$,

$$\langle\, z(T_1, \varphi_1), z(T_2, \varphi_2)\,\rangle = A(T_1, T_2; \lambda_{V_f}(\varphi_1 \otimes \varphi_2)).$$

▶ In short, the inner products of special cycle classes are given by Fourier coefficients of pullbacks of Hilbert-Siegel Eisenstein series.

▶ Also note that such pullbacks have a non-trivial cuspidal component mediated by special values of doubling L-functions.

But one can get more!

▶ Tripling gives a formula for the product itself!

**Formula for products:**
For $n_1 + n_2 + n_3 = m$, and for weight functions $\varphi_i \in S(V(\mathbb{A}_f)^{n_i})$, write

$$E(\begin{pmatrix} \tau_1 & & \\ & \tau_2 & \\ & & \tau_3 \end{pmatrix}, \frac{1}{2}, \lambda_{V_f}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3))$$
$$= \sum_{T_1 \in \mathrm{Sym}_{n_1}(F)_{\geq 0}} \sum_{T_2 \in \mathrm{Sym}_{n_2}(F)_{\geq 0}} \sum_{T_3 \in \mathrm{Sym}_{n_3}(F)_{\geq 0}} A(T_1, T_2, T_3; \lambda_{V_f}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3))\, \boldsymbol{q}_1^{T_1}\, \boldsymbol{q}_2^{T_2}\, \boldsymbol{q}_3^{T_3}.$$

By the Siegel-Weil formula, this series coincides with

the degree of the triple product of generating series

$$\langle \phi_{n_1}(\tau_1, \varphi_1), \phi_{n_2}(\tau_2, \varphi_2), \phi_{n_3}(\tau_3, \varphi_3) \rangle$$
$$= \sum_{T_1 \in \mathrm{Sym}_{n_1}(F)_{\geq 0}} \sum_{T_2 \in \mathrm{Sym}_{n_2}(F)_{\geq 0}} \sum_{T_3 \in \mathrm{Sym}_{n_3}(F)_{\geq 0}} \langle\, z(T_1, \varphi_1) \cdot z(T_2, \varphi_2), z(T_3, \varphi_3)\,\rangle\, \boldsymbol{q}_1^{T_1}\, \boldsymbol{q}_2^{T_2}\, \boldsymbol{q}_3^{T_3}.$$

**Theorem D:** For $n_1 + n_2 + n_3 = m$,

$$\langle\, z(T_1, \varphi_1) \cdot z(T_2, \varphi_2), z(T_3, \varphi_3)\,\rangle = A(T_1, T_2, T_3; \lambda_{V_f}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)).$$

▶ Since the pairing $\langle\ ,\ \rangle$ is non-degenerate pairing on $\mathrm{SC}^\bullet(V)$, this last formula uniquely determines the product

$$z(T_1, \varphi_1) \cdot z(T_2, \varphi_2) \ \in\ \mathrm{SC}^{n_1+n_2}(V).$$

Thus the structure of the ring $\mathrm{SC}^\bullet(V)$ is determined by the Fourier coefficients of triple pullbacks of Hilbert-Siegel Eisenstein series.

## 5. Comparisons

**Comparison isomorphisms.**
Consider two quadratic spaces $V_1$ and $V_2$ over $F$ with.

$$\dim_F V_1 = \dim_F V_2$$

$$\chi = \chi_{V_1} = \chi_{V_2}.$$

The Rallis maps then have the same target:

$$\lambda_{V_i} : S(V_i(\mathbb{A})^n) \longrightarrow I(s_0, \chi), \qquad \lambda_{V_i}(\varphi_i)(g') = \omega_{V_i}(g')\varphi_i(0),$$

$$s_0 = \frac{1}{2}\dim V_i - \rho_n$$

**Definition 5.1.** Schwartz functions $\varphi_1 \in S(V_1(\mathbb{A})^n)$ and $\varphi_2 \in S(V_2(\mathbb{A})^n)$ **match** if

$$\lambda_{V_1}(\varphi_1) = \lambda_{V_2}(\varphi_2) \in I(s_0, \chi).$$

**Basic Observations:**
(i) If $\varphi_1 \in S(V_1(\mathbb{A})^n)$ and $\varphi_2 \in S(V_2(\mathbb{A})^n)$ are matching Schwartz functions, then *the associated Siegel-Eisenstein series coincide,*

$$E(g', s, \lambda_{V_1}(\varphi_1)) = E(g', s, \lambda_{V_2}(\varphi_2)).$$

(ii) For any $n$, the Schwartz form $\varphi_{KM}^{(n)}$ for signature $(p, q)$ with $p + q = m$, $q$ even, and the Gaussian for signature $(m + 2, 0)$ match locally.

► The point is that the Siegel-Eisenstein series is built out of only local data.

These observations yield a lot of non-trivial identities.

▶ In certain cases we get

**Automatic matching:**

Suppose

$$V(\mathbb{A}_f) \xrightarrow{\sim} V'(\mathbb{A}_f).$$

This implies

$$d_+(V) \equiv d_+(V') \mod 2.$$

Then, for all $n$,

$$S(V(\mathbb{A}_f)^n) \xrightarrow{\sim} S(V'(\mathbb{A}_f)^n), \qquad \varphi \;\leftrightarrow\; \varphi' = \rho^n_{V,V'}(\varphi).$$

and this matching is compatible with tensor products.

As a consequence, we have the comparison isomorphism for special cycle rings stated earlier:

**Comparison Theorem.** For such $V$ and $V'$ there is a linear map

$$\rho_{V,V'} : SC^\bullet(V) \longrightarrow SC^\bullet(V')$$

such that, for $\varphi$ and $\varphi'$ matching

$$\rho_{V,V'} : z_V(T,\varphi) \mapsto z_{V'}(T,\varphi').$$

Moreover, this map is **a ring homomorphism** and **an isometry**.

*Proof.* The rings $SC^\bullet(V)$ and $SC^\bullet(V')$ are spanned by the classes $z_V(T,\varphi)$ and $z_{V'}(T,\varphi')$. Suppose that there is a linear relation in $SC^n(V)$

$$\sum_i c_i\, z_V(T_i,\varphi_i) = 0, \qquad \varphi_i \in S(V(\mathbb{A}_f)^n),\ T_i \in \mathrm{Sym}_n(F),\ c_i \in \mathbb{C}.$$

Then,

$$\begin{aligned} 0 &= \langle \sum_i c_i\, z_V(T_i,\varphi_i), z_V(T,\varphi) \rangle \\ &= \sum_i c_i \langle\, z_V(T_i,\varphi_i), z_V(T,\varphi)\, \rangle \\ &= \sum_i c_i\, A(T_i,T;\lambda_{V_f}(\varphi_i\otimes\varphi)) = \sum_i c_i\, A(T_i,T;\lambda_{V_f}(\varphi_i'\otimes\varphi')) \\ &= \ \langle \sum_i c_i\, z_{V'}(T_i,\varphi_i'), z_{V'}(T,\varphi') \rangle \end{aligned}$$

for all pairs $T$ and $\varphi$. Here the Schwartz functions are matching those. Since the inner product on the ring $SC^\bullet(V')$ is non-degenerate by construction, we have

$$\sum_i c_i\, z_{V'}(T_i,\varphi_i') = 0$$

in $SC^n(V')$. Thus the linear map $\rho_{V,V'}$ is well defined.

Isometry and ring homomorphism follow similarly. □

**Another comparison.**

It is tempting to ask what happens when $d_+ = 0$.

It is possible to set up a theory of 'special cycles' that is quite parallel to the $d_+ > 0$ case!

I will skip to the final formulas.

$$\begin{aligned} V_+ &= \text{totally positive definite quadratic space over } F, \\ \dim_F V_+ &= m+2 \\ G_+ &= R_{F/\mathbb{Q}}\mathrm{SO}(V_+) \\ H^\bullet(V_+) &= C_{\mathrm{cont}}(G_+(\mathbb{Q})\backslash G_+(\mathbb{A}_f)) \otimes_{\mathbb{Z}} \mathbb{Z}[\boldsymbol{c}], \qquad \boldsymbol{c}^{m+1} = 0 \\ &= \text{truncated polynomial ring} \\ \deg(\boldsymbol{c}^r) &= \begin{cases} 1 & r = m \\ 0 & \text{otherwise.} \end{cases} \qquad \boldsymbol{z} : G_+(\mathbb{Q})\backslash G_+(\mathbb{A}_f) \longrightarrow \mathbb{C}[\boldsymbol{c}], \\ \deg^{\mathrm{tot}}(\boldsymbol{z}) :&= \int_{G_+(\mathbb{Q})\backslash G_+(\mathbb{A}_f)} \deg(\boldsymbol{z}(g))\, d_f g. \end{aligned}$$

For $\varphi \in S(V_+(\mathbb{A}_f)^n)$, $T \in \mathrm{Sym}_n(F)$, define a special cycle:

$$\begin{aligned} \mathrm{Rep}_+(T,\varphi)(g) :&= \sum_{\substack{x \in V_+(F)^n \\ Q(x) = T}} \varphi(g^{-1}x) \qquad \in C_{\mathrm{cont}}(G_+(\mathbb{Q})\backslash G_+(\mathbb{A}_f)), \\ \boldsymbol{z}(T,\varphi)^\natural :&= \mathrm{Rep}_+(T,\varphi)\, \boldsymbol{c}^n \qquad \in H^n(V_+). \end{aligned}$$

**Product formula!**

$$\boldsymbol{z}(T_1,\varphi_1)^\natural \cdot \boldsymbol{z}(T_2,\varphi_2)^\natural = \sum_{\substack{T \in \mathrm{Sym}_{n_1+n_2}(F)_{\geq 0} \\ T = \begin{pmatrix} T_1 & * \\ {}^t* & T_2 \end{pmatrix}}} \boldsymbol{z}(T, \varphi_1 \otimes \varphi_2)^\natural.$$

So we again get a subring $\mathrm{SC}^\bullet(V_+)^\natural$ of 'special cycles' in the 'cohomology' ring $H^\bullet(V_+)$.
It includes $\mathbb{1}$.

**The reduced ring.**

For special cycles in complementary degrees $n_1$ and $n_2$ with $n_1 + n_2 = m$, the **inner product** is given by

$$\langle\, \boldsymbol{z}(T_1,\varphi_1)^\natural, \boldsymbol{z}(T_2,\varphi_2)^\natural\,\rangle = \int_{G_+(\mathbb{Q})\backslash G_+(\mathbb{A}_f)} \mathrm{Rep}_+(T_1,\varphi_1)(g)\,\mathrm{Rep}_+(T_2,\varphi_2)(g)\,dg.$$

Let

$$\mathrm{SC}^\bullet(V_+) = \mathrm{SC}^\bullet(V_+)^\natural/\mathcal{R}\mathrm{SC}^\bullet(V_+)^\natural$$

We denote the image of $\boldsymbol{z}(T,\varphi)^\natural \in \mathrm{SC}^\bullet(V_+)^\natural$ in $\mathrm{SC}^\bullet(V_+)$ by $\boldsymbol{z}(T,\varphi)$.

**Lemma** (i) For $n \le \frac{1}{2}m$, the map $\mathrm{SC}^n(V_+)^\natural \longrightarrow \mathrm{SC}^n(V_+)$ is an isomorphism.
(ii) On the other hand, for $n = m$, $\mathrm{SC}^m(V_+) = \mathbb{C}\,\boldsymbol{c}^m$ and the map $\mathrm{SC}^m(V_+)^\natural \longrightarrow \mathrm{SC}^m(V_+)$ is given by

$$\boldsymbol{z}^\natural \mapsto \boldsymbol{z} = \deg^{\mathrm{tot}}(\boldsymbol{z}^\natural)\cdot \boldsymbol{c}^m.$$

**Example: The middle dimension:** In particular, if $m = 2n$ is even, then

$$C_{\mathrm{cont}}(G_+(\mathbb{Q})\backslash G_+(\mathbb{A}_f)) \overset{\sim}{\longrightarrow} \mathrm{SC}^n(V_+), \qquad \phi \mapsto \phi\cdot \boldsymbol{c}^n,$$

with the inner product given by ($\star$).

**Another comparison isomorphism.**

**Theorem E:** For a quadratic space $V$ over $F$ with $d_+(V)$ even, let $V_+$ be the associated totally positive definite space with $V(\mathbb{A}_f) \overset{\sim}{\longrightarrow} V_+(\mathbb{A}_f)$. Fix an isometry $\rho_{V,V_+} : V(\mathbb{A}_f) \overset{\sim}{\longrightarrow} V_+(\mathbb{A}_f)$. Then there is a linear map

$$\rho_{V,V_+} : \mathrm{SC}^\bullet(V) \longrightarrow \mathrm{SC}^\bullet(V_+)$$

such that, for $\varphi$ and $\varphi'$ matching

$$\rho_{V,V_+} : z_V(T,\varphi) \mapsto \boldsymbol{z}_{V_+}(T,\varphi').$$

Moreover, this map is a ring homomorphism and an isometry.

► Thus, for $d_+$ even, the reduced special cycle rings are given by the 'combinatorial' construction just described! Note the fundamental role played by the ring

$$C_{\mathrm{cont}}(G_+(\mathbb{Q})\backslash G_+(\mathbb{A}_f))$$

for the associated totally positive definite space $V_+$.

► Do these comparison isomorphism have a motivic origin?

► Is there a way to account for them in terms of automorphic representations, e.g., as a consequence of functoriality among inner forms of $\mathrm{SO}(V)$?

► Is there a 'combinatorial model' in the case where $d_+$ is odd?