Weighted normal bundles and the isotropic embedding theorem

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Based on:

Euler-like vector fields, normal forms, and isotropic embeddings, arXiv:2001.10518

and forthcoming work with Y. Loizides

Review: Euler-like vector fields

Definition (Bursztyn-Lima-M)

A vector field $X \in \mathfrak{X}(M)$ is Euler-like with respect to a submanifold $N \subset M$ if $X|_N = 0$ and the linear approximation

$$X_{[0]} \in \mathfrak{X}(\nu(M, N))$$

is the Euler vector field $\mathcal{E}$.

Lemma (Bursztyn-Lima-M)

If X is Euler-like with respect to $N \subset M$, there is a unique germ of a tubular neighborhood embedding

$$\phi \colon \nu(M, N) \to M$$

such that $\phi^* X = \mathcal{E}$.

Euler-like vector fields $\overset{1-1}{\longleftrightarrow}$ tubular neighborhood embeddings

on level of germs along N

Review: Euler-like vector fields

Application: Let (M, ω) symplectic manifold, $N \subset M$ Lagrangian. Have linear approximation

$$\omega_{[1]} \in \Omega^2(\nu(M, N)).$$

Theorem (Weinstein)

There exists a tubular nbd $\phi\colon \nu(M, N) \to M$ *such that* $\phi^*\omega = \omega_{[1]}$.

Sketch of proof.

1. Choose $\alpha \in \Omega^1(M)$, vanishing along N, with $\mathrm{d}\alpha = \omega$.
2. $X \in \mathfrak{X}(M)$ with $\iota_X\omega = \alpha$ is Euler-like, so defines ϕ with $\phi^*X = \mathcal{E}$.
3. $$\mathcal{L}_{\mathcal{E}}\phi^*\omega = \phi^*\mathcal{L}_X\omega = \phi^*\mathrm{d}\alpha = \phi^*\omega$$

   implies that $\phi^*\omega$ is linear, i.e. equal to $\omega_{[1]}$. □

$\mathcal{Q}$: What about *isotropic* submanifolds $N \subset M$?

Here $\omega_{[1]} \in \Omega^2(\nu(M, N))$ is well-defined, but not symplectic.

Reason: In some directions normal to N, the symplectic form ω vanishes to second order rather than linearly.

Idea: Use approximation with weights.

Weightings

Fix a weight sequence

$$0 \le w_1 \le w_2 \le \cdots \le w_n \le r.$$

For $U \subseteq \mathbb{R}^n$, get filtration by ideals

$$C^\infty(U) = C^\infty(U)_{(0)} \supseteq C^\infty(U)_{(1)} \supseteq \cdots$$

where $C^\infty(U)_{(i)}$ generated by monomials

$$x^s = x_1^{s_1} \cdots x_n^{s_n}, \quad \sum_a s_a w_a \ge i.$$

Definition (Loizides-M)

An order r weighting on a manifold M is given by an atlas, where all transition maps preserve weight filtrations.

Weightings

An order r weighting on M determines a filtration

$$(*) \qquad C^\infty(M) = C^\infty(M)_{(0)} \supseteq C^\infty(M)_{(1)} \supseteq \cdots$$

where

$$\mathcal{I} = C^\infty(M)_{(1)}$$

is the vanishing ideal of a closed submanifold $N \subset M$. If N is given, we speak of a weighting along N.

The case $r = 1$ is trivial weighting, where $C^\infty(M)_{(k)} = \mathcal{I}^k$.

In general, think of $C^\infty(M)_{(k)}$ as vanishing to order k on $N \subset M$ *in the weighted sense*.

Weighted normal bundle

Theorem (Loizides-M)

An order r weighting along $N \subset M$ determines a unique fiber bundle $\nu_{\mathcal{W}}(M,N) \to N$, with an action $t \mapsto \kappa_t$ of $(\mathbb{R}, \cdot)$, such that

- *$N \subset \nu_{\mathcal{W}}(M,N)$ is the fixed point set of the $(\mathbb{R}, \cdot)$-action,*
- $$C^\infty(M)_{(k)}/C^\infty(M)_{(k+1)} = C^\infty(\nu_{\mathcal{W}}(M,N))_{[k]}$$
  the functions homogeneous of degree k.

For $r = 1$, recover $\mathcal{I}^k/\mathcal{I}^{k+1} \cong \Gamma\big(\mathrm{Sym}^k(\nu(M,N)^*)\big) = C^\infty(\nu(M,N))_{[k]}$.

Note: Every $f \in C^\infty(M)_{(k)}$ determines an order k approximation

$$f_{[k]} \in C^\infty(\nu_{\mathcal{W}}(M,N))_{[k]}$$

Likewise for forms, vector fields, etc.

Weighted normal bundle

Remark

The weighted normal bundle $\nu_{\mathcal{W}}(M,N)$ has an alternative description as a subquotient of

$$T_r M = J_0^r(\mathbb{R}, M),$$

the r-th tangent bundle.

For $r = 1$, this is $\nu(M,N) = TM|_N / TN$.

For $r > 1$, $\nu_{\mathcal{W}}(M,N)$ is not a vector bundle, but is a *graded bundle* in the sense of Grabowski-Rotkievicz.

Weighted Euler-like vector fields

Let $N \subset M$ with order r weighting $\rightsquigarrow C^\infty(M) \supseteq C^\infty(M)_{(1)} \supseteq \cdots$.

Definition

$X \in \mathfrak{X}(M)$ is weighted Euler-like if it has filtration degree 0, with weighted homogeneous approximation

$$X_{[0]} = \mathcal{E},$$

the Euler vector field of $\nu_{\mathcal{W}}(M, N)$.

Theorem

A weighted Euler-like vector field X determines a unique weighted tubular neighborhood embedding

$$\phi \colon \nu_{\mathcal{W}}(M, N) \to M$$

such that $\phi^* X = \mathcal{E}$ (the Euler vf).

Weightings for $r = 2$

An order $r = 2$ weighting along $N \subset M$ is equivalent to a subbundle
$$F \subseteq \nu(M, N).$$

The filtration is generated by

$$C^\infty(M)_{(1)} = \mathcal{I} = \text{ vanishing ideal of } N$$
$$C^\infty(M)_{(2)} = \mathcal{J} = \{f \in \mathcal{I} \colon \mathrm{d}f \text{ vanishes on } \widetilde{F}\}$$

So: any subbundle $F \subseteq \nu(M, N)$ determines a weighted normal bundle
$$\nu_{\mathcal{W}}(M, N) \to N.$$

The isotropic embedding theorem

Let (M, ω) be symplectic, $N \subset M$ isotropic. Have

$$TN^\omega / TN \subset \nu(M, N).$$

- Get a weighting of order $r = 2$, and corresponding $\nu_{\mathcal{W}}(M, N) \to N$.
- $\omega \in \Omega^2(M)_{(2)} \quad \rightsquigarrow \quad \omega_{[2]} \in \Omega^2(\nu_{\mathcal{W}}(M, N))_{[2]}$, symplectic.
- Choose $\alpha \in \Omega^1(M)_{(2)}$ with $\mathrm{d}\alpha = \omega_{(2)}$, define X by $\iota_X \omega = 2\alpha$.
- X is weighted Euler-like, so get $\phi \colon \nu_{\mathcal{W}}(M, N) \to M$ with
$$\phi^* X = \mathcal{E}.$$
- $\mathcal{L}_{\mathcal{E}} \phi^* \omega = \phi^* \mathcal{L}_X \omega = 2\omega$ implies $\phi^* \omega = \omega_{[2]}$.

The isotropic embedding theorem

In summary:

1) For every isotropic submanifold N of (M, ω) there is a canonically defined local model

$$(\nu_{\mathcal{W}}(M, N), \omega_{[2]})$$

2) There is a weighted tubular nbd embedding $\nu_{\mathcal{W}}(M, N) \to M$, preserving symplectic forms.

This is a (small) improvement of *Weinstein's isotropic embedding theorem*, where the construction of the 2-form on the local model

$$TN \oplus TN^{\omega}/TN$$

involves choices.

Outlook

Concluding remarks:

- The theory of weightings comes with a theory of *weighted deformation spaces* and *weighted (real) blow-ups*.
- One can generalize further to 'multi-weightings'.
- Other applications include *filtered manifolds* (Morimoto, Melin); these have been much studied in index theory lately (Choi-Ponge, van Erp, Yuncken, Haj-Higson, Dave-Haller, Mohsen)
- More generally, examples arise from *singular Lie filtrations*:

$$\mathfrak{X}(M) = \mathcal{H}_{-r} \supseteq \mathcal{H}_{-r+1} \supseteq \cdots \supseteq \mathcal{H}_0, \qquad [\mathcal{H}_i, \mathcal{H}_j] \subseteq \mathcal{H}_{i+j}$$

(each $\mathcal{H}_i$ locally finitely generated); every leaf of the singular foliation of $\mathcal{H}_0$ has a canonical weighting.

Thanks!