Bott Canonical Basis?

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Canonical basis

$K \circlearrowright V \qquad$ unitary irrep of compact Lie group K

$$V = \bigoplus_{\alpha \in \mathfrak{t}^*_{\mathbb{Z}}} V_\alpha \qquad \text{decomposition into weights spaces for } T \subset K$$

Goal: decompose $V_\alpha = \underbrace{\mathbb{C}_\alpha \oplus \ldots \oplus \mathbb{C}_\alpha}_{\mathrm{mult}(\alpha)}$ *"canonically"*.

We achieve this using a complex analytic big torus action, modulo a conjectural cohomology-vanishing condition.

Big torus action

$$(S^1)^{\text{big}} \circlearrowright \begin{array}{c} L \\ \downarrow \\ M \end{array} \text{ holomorphic, } M \text{ connected and has fixed points,}$$

then

$\Gamma_{\text{hol}}(M, L)$ splits into one dimensional weights spaces.

$\text{big} := \dim_{\mathbb{C}} M$

Geometric models for representations

$V = \Gamma_{\text{hol}}(M, L)$.

Borel-Weil: $M =$ flag manifold G/B, $\quad L_\lambda = G \times_B \mathbb{C}_{-\lambda}$

Demazure: $M =$ Bott-Samelson manifold

Bott's idea: use $(S^1)^{\text{big}} \circlearrowright \begin{array}{c} L \\ \downarrow \\ M \end{array}$ on Bott-Samelson manifold.

But: holomorphic for a *different* complex structure ("Bott tower").

$B \circlearrowright \mathbb{C}_{-\lambda}$ through $\psi_0 \colon B \to H$. e.g., for $\mathrm{SL}(3,\mathbb{C})$, $\psi_0 \colon \begin{bmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{bmatrix} \mapsto \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Bott-Samelson manifolds

$\alpha_{i_1}, \ldots, \alpha_{i_n}$ sequence of simple positive roots. For each α_{i_j}:

P_{i_j} minimal nontrivial parabolic; e.g., $\begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{bmatrix}$ in $SL(3, \mathbb{C})$

K_{i_j} maximal compact in P_{i_j}; e.g., $\begin{bmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{bmatrix}$ in SU(3)

T_{i_j} the centre of K_{i_j}; e.g., $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$

Real version (Bott&Samelson, 1950s):

$$K_{i_1} \times_T K_{i_2} \times_T \ldots \times_T K_{i_n}/T \xrightarrow{\text{multiplication}} K/T.$$

Left action of $(T \times \ldots \times T)_{\text{eff}} = T \times_{T_{i_1}} T \times_{T_{i_2}} \ldots \times_{T_{i_{n-1}}} T/T_{i_n}$.

Complex version (Demazure, 1970s):

$$P_{i_1} \times_B P_{i_2} \times_B \ldots \times_B P_{i_n}/B \xrightarrow{\text{multiplication}} G/B.$$

Also with associated line bundles.

Bott-Samelson manifold: $(P_{i_1} \times \cdots \times P_{i_n})/B^n$, where

$$(p_1, p_2, \ldots, p_n) \cdot (b_1, \ldots, b_n)$$
$$= (p_1 b_1\,,\ b_1^{-1} p_2 b_2\,,\ \ldots\,,\ b_{n-1}^{-1} p_n b_n).$$

Bott tower: $(P_{i_i} \times \cdots \times P_{i_n})/B^n$, where

$$(p_1, p_2, \ldots, p_n) \cdot (b_1, \ldots, b_n)$$
$$= (p_1 b_1\,,\ \psi_0(b_1)^{-1} p_2 b_2\,,\ \ldots\,,\ \psi_0(b_{n-1})^{-1} p_n b_n);$$

$\psi_0 \colon B \to H$ projection from the Borel to the Cartan.
Has left action of $(T \times \ldots \times T)_{\text{eff}}$.

Bott Samelson family

The Bott tower and the Bott-Samelson manifold fit into a family

$$X_{\mathbf{t}} := (P_{i_i} \times \cdots \times P_{i_n})/B^n$$

for $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{C}^n$, where

$$\begin{aligned}(p_1, p_2, \ldots, p_n) \cdot (b_1, \ldots, b_n) \\ = (p_1 b_1\,,\ \psi_{t_2}(b_1)^{-1} p_2 b_2\,,\ \ldots\,,\ \psi_{t_n}(b_{n-1})^{-1} p_n b_n),\end{aligned}$$

where $\psi_t \colon B \to B$ for $t \in \mathbb{C}$ is

$$\psi_t = \begin{cases} \text{conjugation by } S(t) & \text{when } t \neq 0 \\ \text{the projection } B \to H & \text{when } t = 0 \end{cases}$$

for appropriate one-parameter subgroup $S \colon \mathbb{C}^\times \to H$

e.g., $t \mapsto \operatorname{diag}(t, 1, t^{-1})$, $\quad \psi_t : \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \mapsto \begin{bmatrix} a & tb & t^2 c \\ 0 & d & te \\ 0 & 0 & f \end{bmatrix}$

Also with associated line bundles.

Equivariant families

$$(\mathbb{C}^\times)^n \circlearrowright \mathfrak{L} \longrightarrow \mathfrak{X} \xrightarrow{\pi} \mathbb{C}^n .$$

$X_\mathbf{0} := \pi^{-1}(0,\dots,0); \quad X_\mathbf{1} := \pi^{-1}(1,\dots,1).$

Condition 1: the action of each $(\mathbb{C}^\times)_{j^{\text{th}}}$ on $X_\mathbf{0}$ has a fixed point with all isotropy weights ≤ 0.

Condition 2: $H^{\geq 1}(\mathfrak{X}, \mathcal{O}_\mathfrak{L}) = \{0\}$.

- Obtain a filtration $\{F_{\vec{\ell}}\}_{\vec{\ell}\in\mathbb{Z}^n}$ of $V := \Gamma_{\text{hol}}(X_\mathbf{1}, \mathfrak{L}|_{X_\mathbf{1}})$ and a map

$$V_{\text{graded}} := \bigoplus_{\vec{\ell}} F_{\vec{\ell}}/F_{>\vec{\ell}} \longrightarrow \Gamma_{\text{hol}}(X_\mathbf{0}, \mathfrak{L}|_{X_\mathbf{0}})$$

  whose restriction to each summand is one-to-one.
- If $X_\mathbf{0}$ is toric, each summand is one dimensional.
- $V_{\text{graded}} \to V$ is a linear isomorphism.

Putting everything together

We define a left $(\mathbb{C}^\times)^n$ action with which the Bott-Samelson family

$$\mathfrak{X} = (\mathbb{C}^n \times P_{i_1} \times \cdots \times P_{i_n})/B^n \xrightarrow{\pi} \mathbb{C}^n$$

becomes an equivariant family with $X_{\mathbf{0}}$ the Bott tower and $X_{\mathbf{1}}$ the Bott-Samelson manifold.

If $\alpha_{i_1}, \ldots, \alpha_{i_n}$ corresponds to a reduced expression of the longest element of the Weyl group, and we take the line bundle $\mathfrak{L} = (\mathbb{C}^n \times P_{i_1} \times \cdots \times P_{i_n}) \times_{B^n} \mathbb{C}_{0,\ldots,0,-\lambda}$, then $V := \Gamma_{\mathrm{hol}}(X_{\mathbf{1}}, \mathfrak{L}|_{X_{\mathbf{1}}})$ provides a geometric model for the (restriction to B of the) irreducible representation of G with highest weight λ.

If $H^{\geq 1}(\mathfrak{X}, \mathcal{O}_{\mathfrak{L}}) = \{0\}$, we obtain a decomposition of V into one dimensional spaces that refines its decomposition into the T weights spaces.

Thank you!