Generalized hyperpolygons, meromorphic Higgs bundles on curves, and integrability

June 6, 2020

Lie Theory and Integrable Systems in Symplectic and Poisson Geometry

Generalized hyperpolygons, meromorphic Higgs bundles on curves, and integrability

Joint work with Laura Schaposnik

arXiv:2001.06911

June 6, 2020
Lie Theory and Integrable Systems in Symplectic and Poisson Geometry

1. Flags varieties to hyperpolygons

2. Hyperpolygons to Higgs bundles

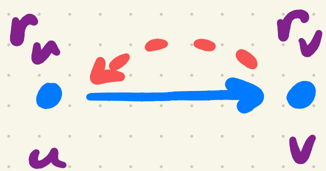
3. Integrability

1. Flags varieties to hyperpolygons

$$\underset{u}{\overset{r_u}{\bullet}} \longrightarrow \underset{v}{\overset{r_v}{\bullet}}$$

$$\mathrm{Rep}\left(\begin{array}{ccc} r_u & & r_v \\ \bullet & \longrightarrow & \bullet \\ u & & v \end{array}\right) = \mathrm{Hom}(\mathbb{C}^{r_u}, \mathbb{C}^{r_v})$$

Nakajima quiver



$$\mathrm{Rep}\left(\begin{tikzcd} \overset{r_u}{\underset{u}{\bullet}} \arrow[r] & \overset{r_v}{\underset{v}{\bullet}} \arrow[l, dashed, bend right] \end{tikzcd}\right) = T^*\mathrm{Hom}(\mathbb{C}^{r_u}, \mathbb{C}^{r_v})$$

$$\mathrm{Rep}\Big(\underset{u}{\overset{r_u}{\bullet}} \underset{x}{\overset{y}{\rightleftarrows}} \underset{v}{\overset{r_v}{\bullet}} \Big) = T^{*}\,\mathrm{Hom}(\mathbb{C}^{r_u}, \mathbb{C}^{r_v})$$

$$\cong$$

$$\underbrace{\mathrm{Hom}(\mathbb{C}^{r_u}, \mathbb{C}^{r_v})}_{x} \oplus \underbrace{\mathrm{Hom}(\mathbb{C}^{r_v}, \mathbb{C}^{r_u})}_{y}$$

To each quiver Q, we can associate 2 functions

$$\mu : T^* \mathrm{Rep}(Q) \longrightarrow \bigoplus_v \mathfrak{u}(r_v)^*$$

$$\nu : T^* \mathrm{Rep}(Q) \longrightarrow \bigoplus_v \mathfrak{gl}(r_v)^*$$

Can build μ, ν node-by-node

For $\begin{tikzcd} u \arrow[r, "x"'] & v \arrow[l, bend right, dashed, "y"'] \end{tikzcd}$:

$$\mu_u(x, y) = \overbrace{x^* x - y y^*}^{\mathfrak{u}(r_u)^*}$$

$$\nu_u(x, y) = \underbrace{xy}_{\mathfrak{gl}(r_u)^*}$$

From Q and its moment maps, we can construct an (affine) hyperkähler variety, the Nakajima quiver variety of Q

From Q and its moment maps, we can construct an (affine) hyperkähler variety, the Nakajima quiver variety of Q

$$T^*\mathrm{Rep}(Q) /\!\!/ G \,, \quad G = \prod_v GL(r_v, \mathbb{C})$$

From Q and it's moment maps, we can construct an (affine) hyperkähler variety, the Nakajima quiver variety of Q

$$T^*\mathrm{Rep}(Q) /\!\!/\!\!/_{\alpha} G, \quad G = \prod_v GL(r_v, \mathbb{C})$$

From Q and its moment maps, we can construct an (affine) hyperkähler variety, the Nakajima quiver variety of Q

$$\mu^{-1}(\alpha) \cap \nu^{-1}(0) \big/ G \,, \quad G = \prod_v U(r_v) \big/ \pm 1$$

$$\alpha \in Z(\mathfrak{g}^*)$$

From Q and its moment maps, we can construct an (affine) hyperkähler variety, the Nakajima quiver variety of Q

compact!

$$\mu^{-1}(\alpha) \cap \nu^{-1}(0) \Big/ G, \quad G = \prod_v U(r_v) \Big/ \pm 1$$

$$\alpha \in Z(\mathfrak{g}^*)$$

E.g. $\begin{tikzcd} \underset{r}{\bullet} \arrow[r, dashed, bend left] & \cdots \arrow[l] \arrow[r, dashed, bend left] & \underset{r_3}{\bullet} \arrow[l] \arrow[r, dashed, bend left] & \underset{r_2}{\bullet} \arrow[l] \arrow[r, dashed, bend left] & \underset{1}{\bullet} \arrow[l] \end{tikzcd}$

Q is an A-type quiver

Nakajima quiver variety is

$$\underbrace{T^*\mathcal{F}_{\underline{r}}}_{\text{partial flag variety}} \qquad \underline{r} \subseteq [r] = \{1, 2, \dots, r\}$$

$$\#(\underline{r}) = m$$

E.g.
$$\begin{tikzcd} r \arrow[r, dashed, bend left, "y_{m-1}"] & \cdots \arrow[l, "x_{m-1}"] \arrow[r, dashed, bend left] & r_3 \arrow[l] \arrow[r, dashed, bend left, "y_2"] & r_2 \arrow[l, "x_2"] \arrow[r, dashed, bend left, "y_1"] & 1 \arrow[l, "x_1"] \end{tikzcd}$$

Q is an A-type quiver

Nakajima quiver variety is

$$T^*\underbrace{\mathcal{F}_{\underline{r}}}_{\text{partial flag variety}} \qquad \underline{r} \subseteq [r] = \{1, 2, \dots, r\}$$

$$G = \prod_{\underline{r}\setminus\{r\}} U(r_i) \Big/ \pm 1$$

E.g.

$$\begin{tikzcd}
r \arrow[r, dashed, bend left, "y_{m-1}"] & \cdots \arrow[l, "x_{m-1}"] \arrow[r, dashed, bend left] & r_3 \arrow[l] \arrow[r, dashed, bend left, "y_2"] & r_2 \arrow[l, "x_2"] \arrow[r, dashed, bend left, "y_1"] & 1 \arrow[l, "x_1"]
\end{tikzcd}$$

Q is an A-type quiver

Nakajima quiver variety is

$$\underbrace{T^*\mathcal{F}_{\underline{r}}}_{\text{partial flag variety}} \qquad Z(\mathfrak{g}^*) = \bigoplus_{i=1}^{m-1} \mathfrak{u}(r_i)^*$$

$$\rotatebox{90}{$\in$}$$

$$(\alpha, 0, \dots, 0),\ \alpha \in \mathbb{R}$$

$T^*\mathcal{F}_{\underline{r}}$ has $U(r)$ action

w/ complex moment map

$$\nu_m(x_{m-1}, y_{m-1}) = x_{m-1}y_{m-1}$$

$$\operatorname{im}(\nu_m) \subset \operatorname{Nilp}(\mathfrak{gl}(r,\mathbb{C}))$$

$\underline{r} = [r]$ $\Rightarrow$ ν_m Springer resolution of $\mathfrak{sl}(r,\mathbb{C})$

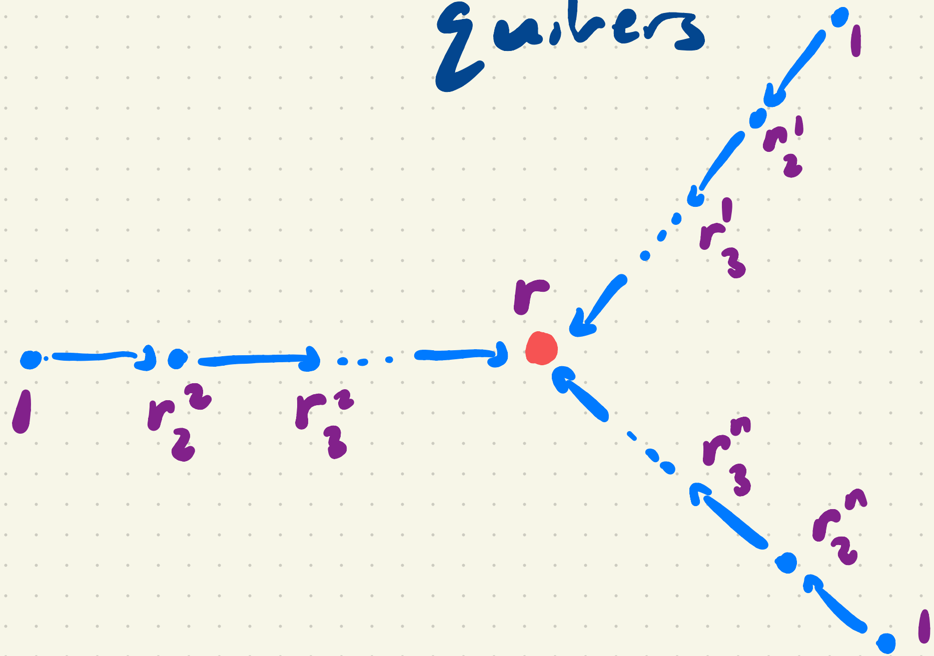
(complete)

Want to use the $U(r)$ action in defining a quiver variety:

interlace n-many A-type quivers

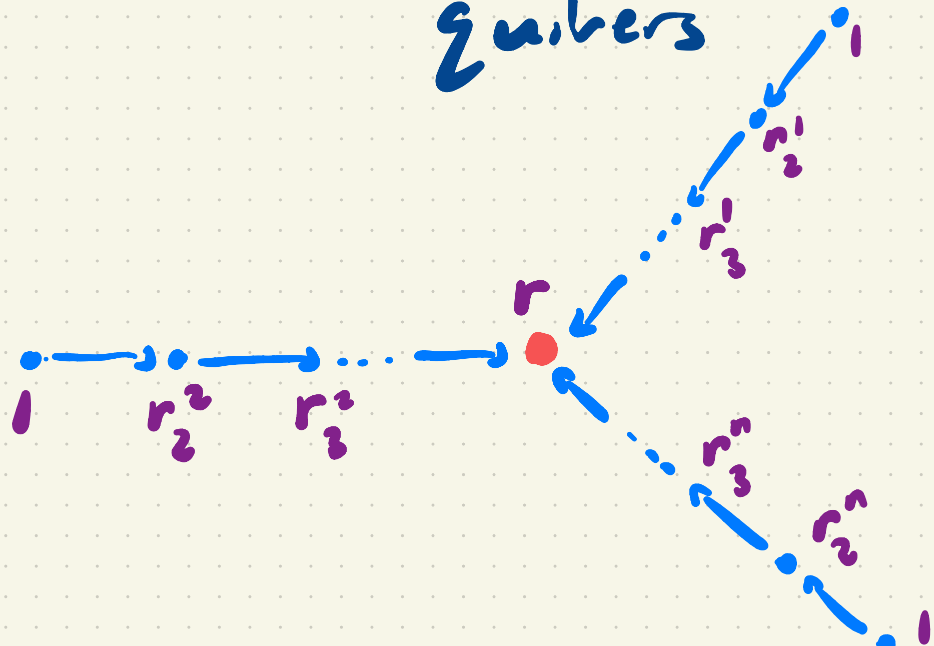
Want to use the $U(r)$ action in defining a quiver variety:

interlace n-many A-type quivers



Want to use the $U(r)$ action in defining a quiver variety:

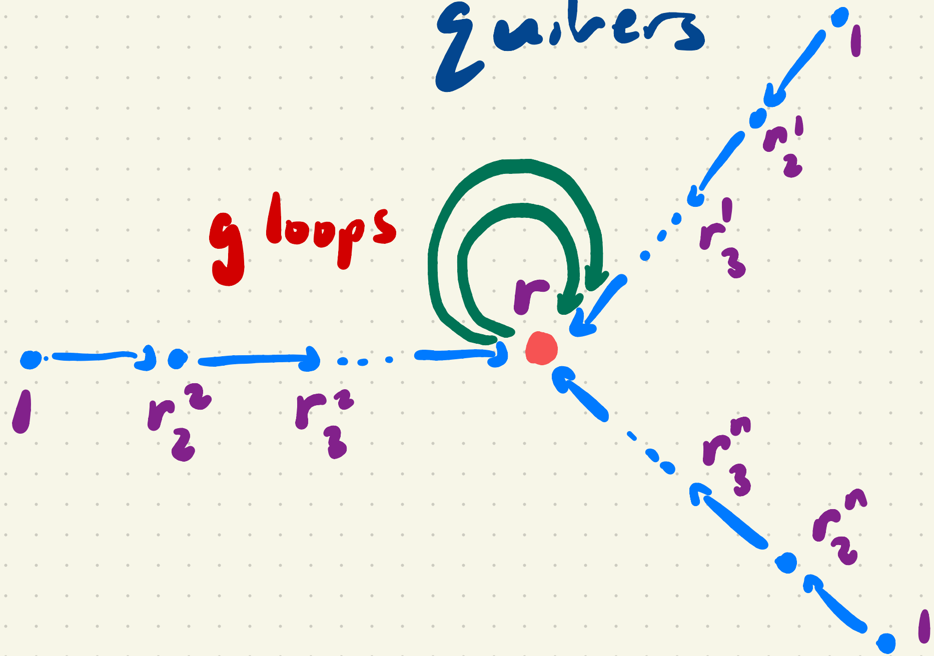
interlace n-many A-type quivers



star-shaped quiver

Want to use the $U(r)$ action in defining a quiver variety:

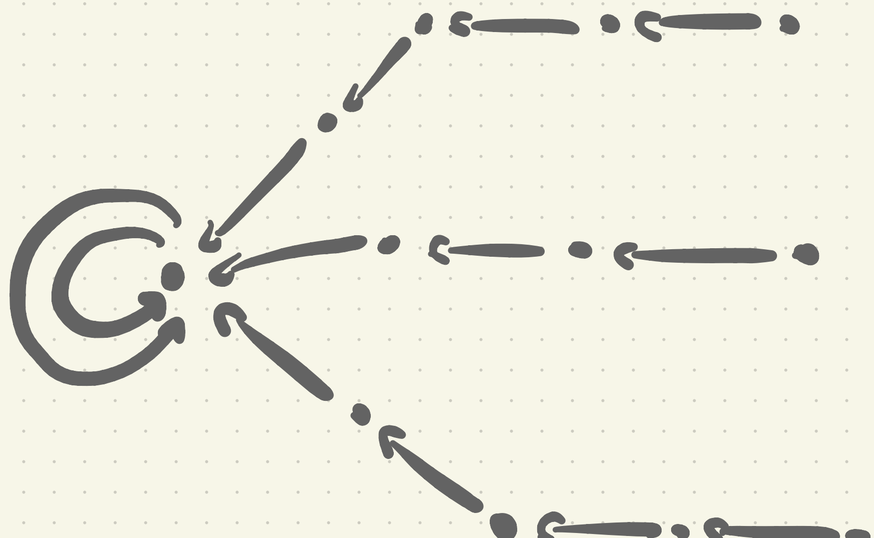
interlace n-many A-type quivers



comet-shaped quiver

Want to use the $U(r)$ action in defining a quiver variety:

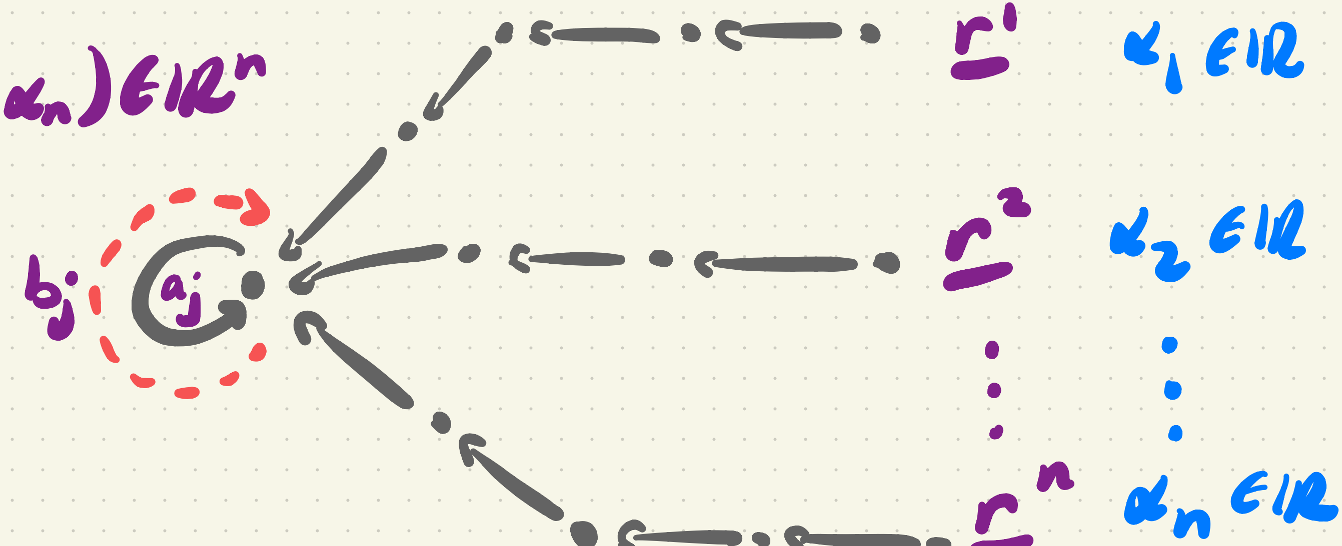
interlace n-many A-type quivers



Want to use the $U(r)$ action in defining a quiver variety:

interlace n-many A-type quivers

$\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$



Node r is the central node

Restrict to $SU(r)$ action at the central node

Nakajima quiver variety of the comet Q is

$$T^*\mathcal{F}_{\underline{r}^1} \times \cdots \times T^*\mathcal{F}_{\underline{r}^n} \times T^*\mathfrak{sl}(r,\mathbb{C})^g /\!/\!/ SU(r)$$

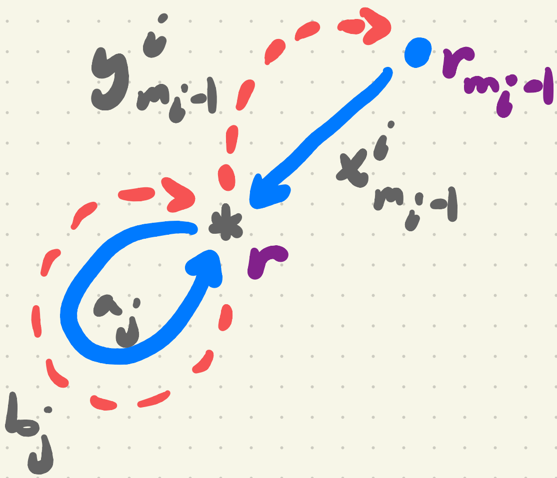
Node r is the central node

Restrict to $SU(r)$ action at the central node

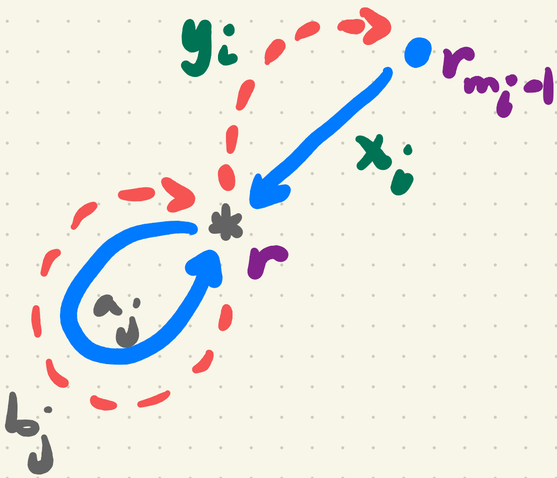
Nakajima quiver variety of the comet Q is

$$T^*\mathcal{F}_{\underline{r}^1} \times \cdots \times T^*\mathcal{F}_{\underline{r}^n} \times T^*\mathfrak{sl}(r,\mathbb{C})^g /\!/\!/_{\underline{0}}\, SU(r)$$

$\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$

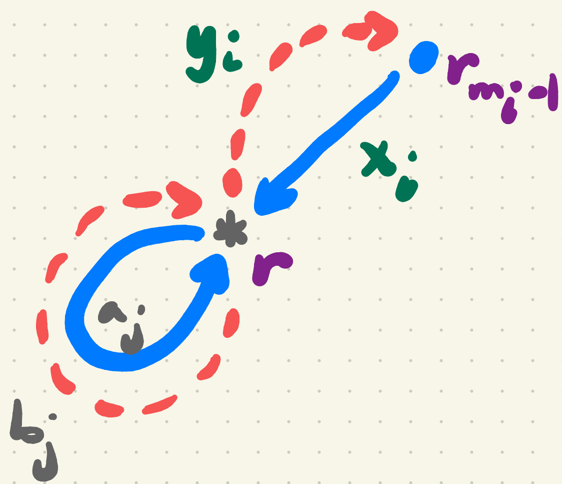


$y^i_{m_i-1}$
r_{m_i-1}
$x^i_{m_i-1}$
r
a^i_j
b^i_j



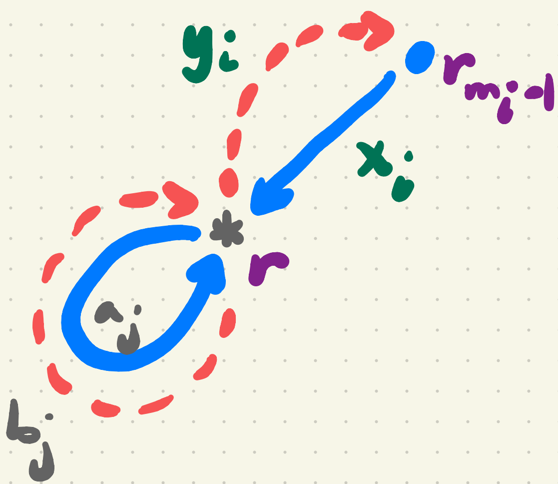
y_i
r_{mid}
x_i
r
a_j
b_j

$$\mu_*(x, y, a, b) = \sum_{i=1}^{n} \left(x_i x_i^* - y_i^* y_i \right)_0 + \sum_{j=1}^{g} [a_j, a_j^*] + [b_j, b_j^*]$$



$$\nu_*(x, y, a, b) = \sum_{i=1}^{n} \left(x_i y_i \right)_0 + \sum_{j=1}^{g} [a_j, b_j]$$

$$\mu_*(x, y, a, b) = \sum_{i=1}^{n} \left(x_i x_i^* - y_i^* y_i \right)_0 + \sum_{j=1}^{g} [a_j, a_j^*] + [b_j, b_j^*]$$



$$\| \quad 0 \quad \|$$

$$\nu_*(x, y, a, b) = \sum_{i=1}^{n} \left(x_i y_i \right)_0 + \sum_{j=1}^{g} [a_j, b_j]$$

Quotient $\mathcal{X}^{g}_{\underline{r}^1, \ldots, \underline{r}^n}(\underline{\alpha}) = \dfrac{\mu_*^{-1}(0) \cap \nu_*^{-1}(\alpha)}{SU(r)}$

is hyperkähler variety w/

$$\dim_{\mathbb{C}} = 2\left(\sum_{i=1}^{n} \dim \mathcal{F}_{\underline{r}^i} + (g-1)(r^2-1)\right)$$

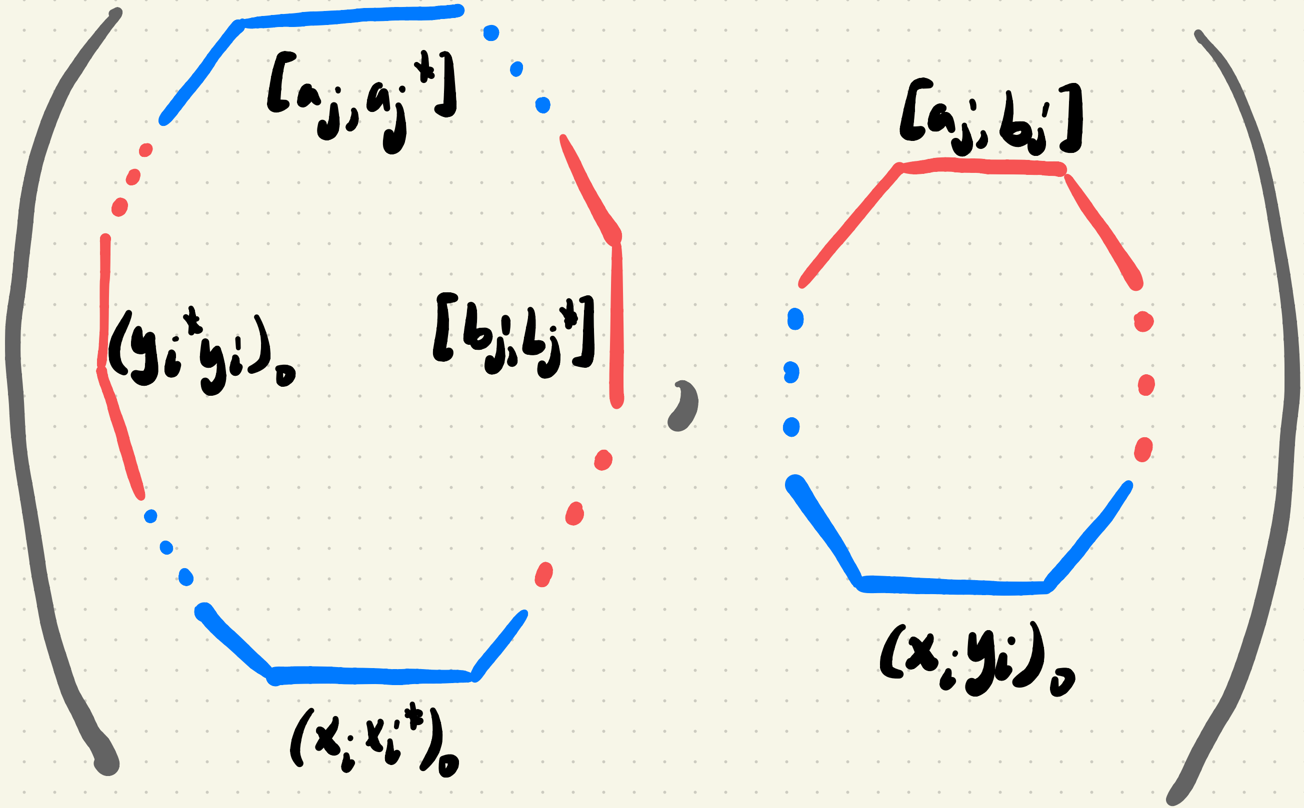
Quotient $\mathcal{X}^{g}_{\underline{r}^1,\dots,\underline{r}^n}(\underline{\alpha}) = \dfrac{\mu_*^{-1}(0)\cap\nu_*^{-1}(0)}{SU(r)}$

is hyperkähler variety w/

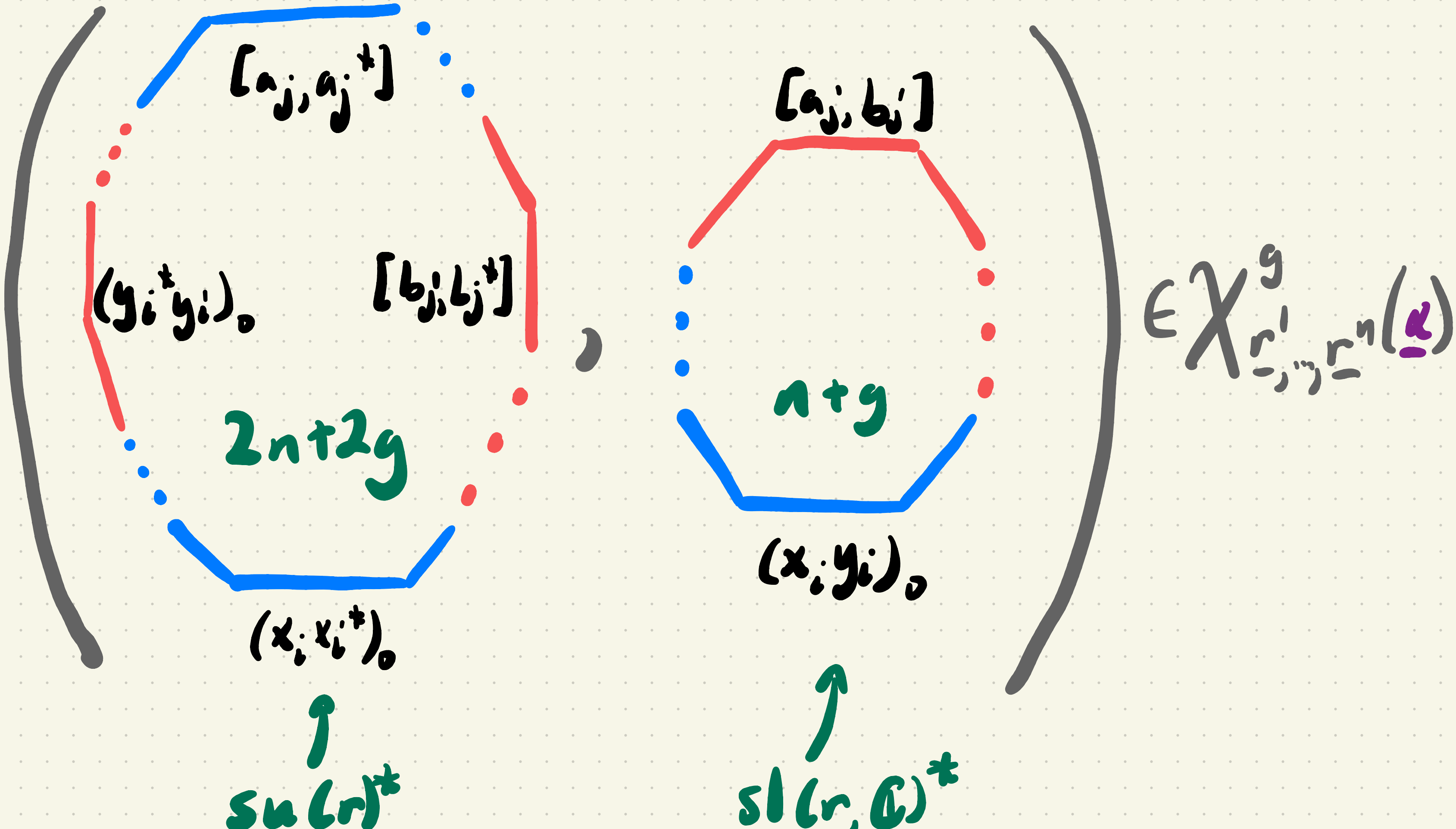
$$\dim_{\mathbb{C}} = 2\left(\sum_{i=1}^{n} \dim \mathcal{F}_{\underline{r}^i} + (g-1)(r^2-1)\right)$$

E.g. when all flags complete,

$$\dim_{\mathbb{C}} = nr(r-1) + 2(g-1)(r^2-1)$$



$\in X^{g}_{\underline{r}',\dots,\underline{r}''}(\underline{\alpha})$


$[a_j, a_j^*]$
$(y_i^* y_i)_0$
$[b_j, b_j^*]$
$2n+2g$
$(x_i x_i^*)_0$
$su(r)^*$
$[a_j, b_j]$
$n+g$
$(x_i y_i)_0$
$sl(r,\mathbb{C})^*$
$\in X^g_{\underline{r}',\dots,\underline{r}''}(\underline{\alpha})$

$X^{g}_{\underline{r}^1, \dots, \underline{r}^n}(\underline{\alpha})$ is the moduli space of (generalized) hyperpolygons of length $\underline{\alpha}$

Konno, Harada-Proudfoot, Godinho-Mandini, Fisher-R

2. Hyperpolygons to Higgs bundles

Nakajima quiver varieties are finite-dim'l analogues of Hitchin systems

$$\left(\sum_{i=1}^{n} \left(x_i x_i^* \right)_0 + \sum_{j=1}^{g} [a_j, a_j^*] \right) + \left(- \sum_{i=1}^{n} \left(y_i^* y_i \right)_0 + \sum_{j=1}^{g} [b_j, b_j^*] \right) = 0$$

$$\sum_{i=1}^{n} \left(x_i y_i \right)_0 = - \sum_{j=1}^{g} [a_j, b_j]$$

$$\left(\sum_{i=1}^{n}\left(x_i\, x_i^*\right)_0 + \sum_{j=1}^{g}[a_j, a_j^*]\right) + \left(-\sum_{i=1}^{n}\left(y_i^*\, y_i\right)_0 + \sum_{j=1}^{g}[b_j, b_j^*]\right) = 0$$

$$\prod_{j=1}^{g}[a_j, b_j]$$

linearization

character variety

$$\sum_{i=1}^{n}\left(x_i\, y_i\right)_0 = -\sum_{j=1}^{g}[a_j, b_j]$$

$$\left(\sum_{i=1}^{n}\left(x_i x_i^*\right)_0 + \sum_{j=1}^{g}[a_j, a_j^*]\right) + \left(-\sum_{i=1}^{n}\left(y_i^* y_i\right)_0 + \sum_{j=1}^{g}[b_j, b_j^*]\right) = 0$$

$$F(A) \quad + \quad \phi \wedge \phi^* \quad = 0$$

$$\sum_{i=1}^{n}\left(x_i y_i\right)_0 = -\sum_{j=1}^{g}[a_j, b_j]$$

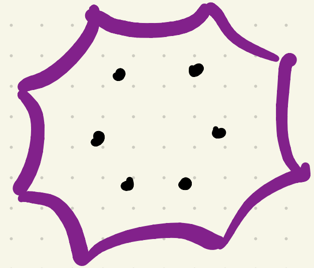
$$\left(\sum_{i=1}^{n}\left(x_i\, x_i^*\right)_0 + \sum_{j=1}^{g}[a_j, a_j^*]\right) + \left(-\sum_{i=1}^{n}\left(y_i^*\, y_i\right)_0 + \sum_{j=1}^{g}[b_j, b_j^*]\right) = 0$$

$$F(A) \qquad + \qquad \phi \wedge \phi^* \qquad = 0$$

$$\bar{\partial}_A \phi \qquad \text{holomorphic}$$

$$\sum_{i=1}^{n}\left(x_i\, y_i\right)_0 = -\sum_{j=1}^{g}[a_j, b_j]$$

$$D = \sum z_i$$

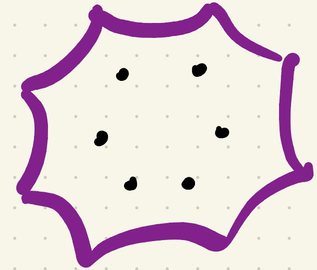


4g-gon

$\subset H$ ($\mathbb{C}$ if $g = 0, 1$)

$$\phi(z) = \sum_{i=1}^{n} \frac{x_i \, y_i}{z - g_z(z_i)} \, dz$$

$$D = \sum z_i$$



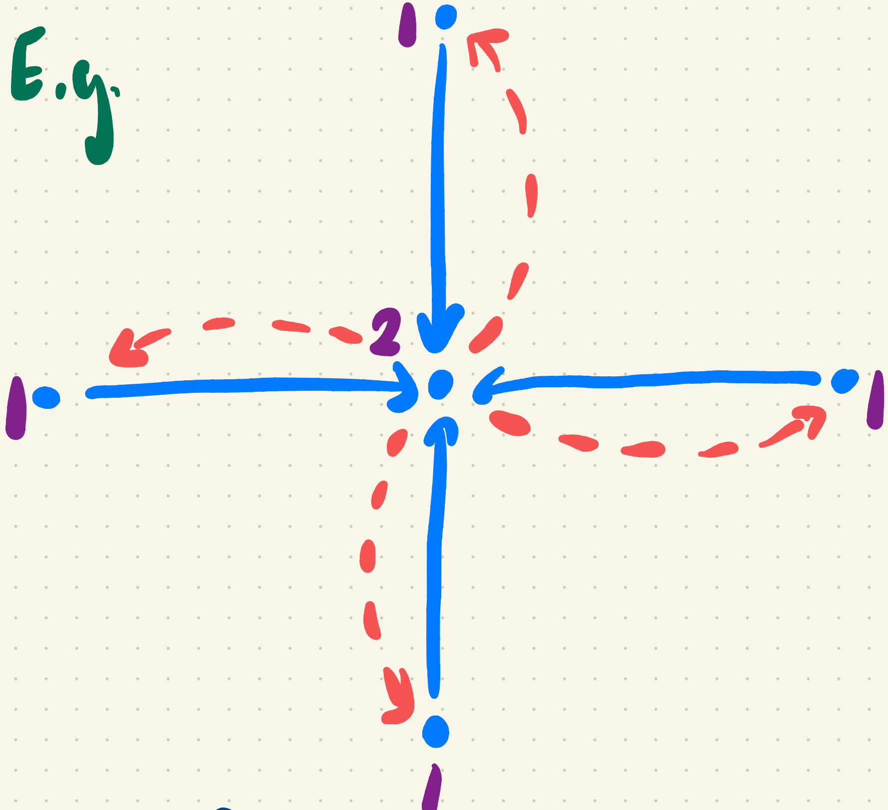
4g-gon

$\subset \mathbb{H}$ ($\mathbb{C}$ if $g=0,1$)

$$\phi(z) = \sum_{i=1}^{n} \frac{x_i \, y_i}{z - g_z(z_i)} \, dz$$

Higgs field for trivial rk-r bundle on $X = \mathbb{H}/\Gamma$ ($\mathbb{C}/\Lambda$ $g=1$, $\mathbb{P}^1$ $g=0$)

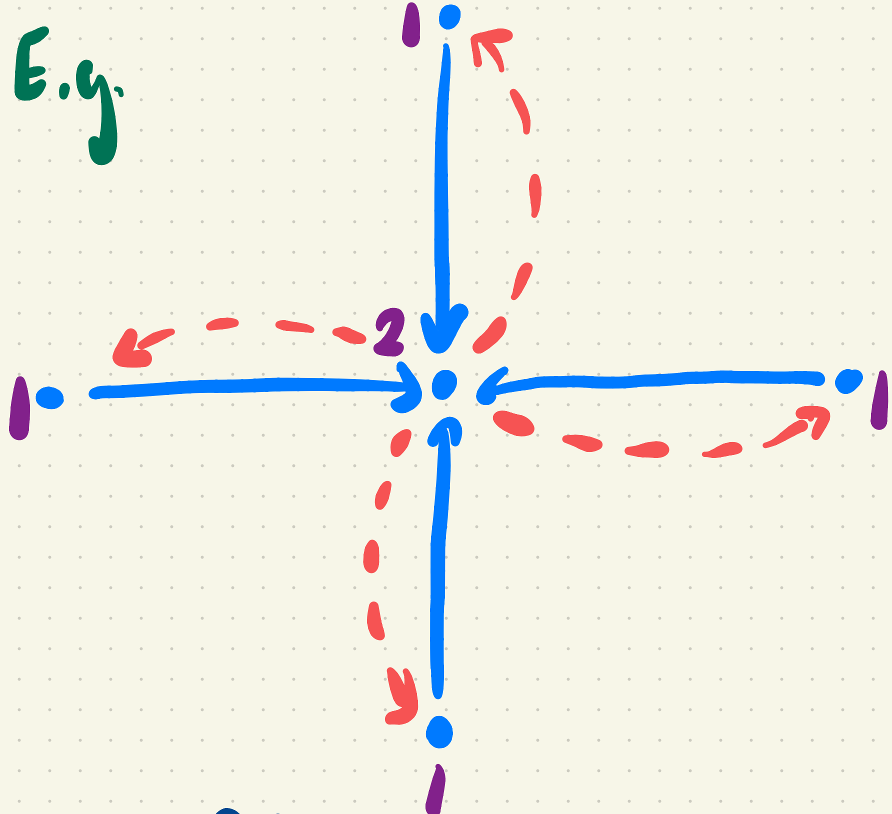
E.g.



$Q = \tilde{D}_4$

$\chi^{\circ}_{[2],[2],[2],[2]}(\underline{\alpha})$

E.g.



$Q = \tilde{D}_4$

$\mathcal{X}^{\circ}_{[2],[2],[2],[2]}(\underline{\alpha})$

is K3 surface

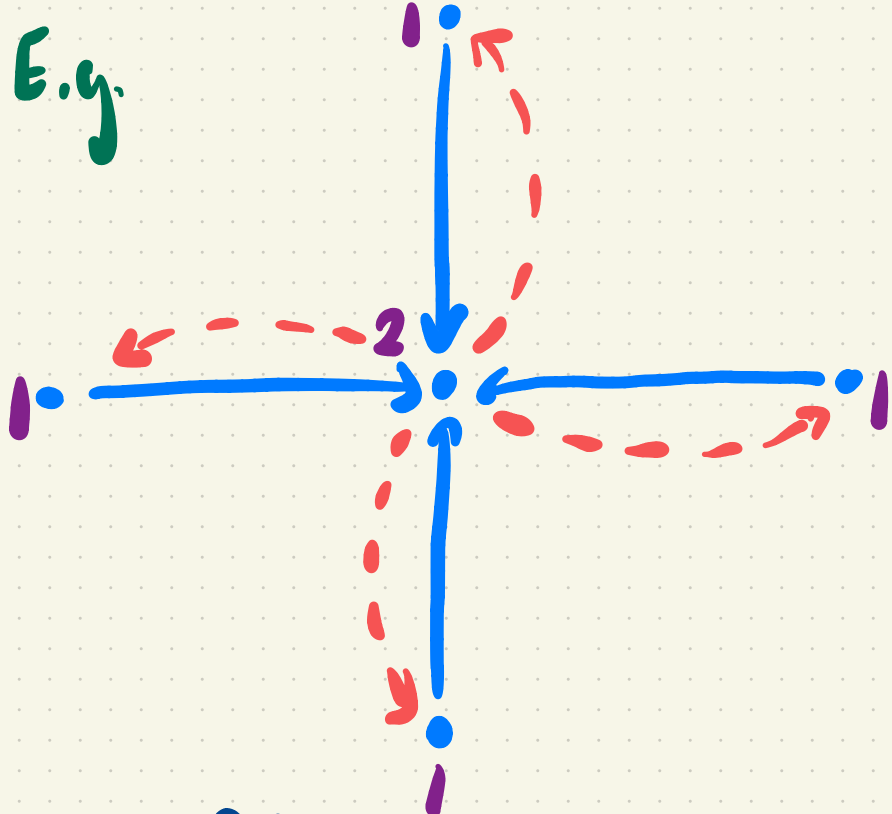
w/ complete ALE metric

Embeds into Hitchin

System on $\mathbb{P}^1 \setminus \{z_1, \dots, z_4\}$

(also surface, but ALG)

E.g.



$Q = \tilde{D}_4$

$$X^{\circ}_{[2],[2],[2],[2]}(\underline{\alpha})$$

is K3 surface w/ complete ALE metric

Embeds into Hitchin System on $\mathbb{P}^1 \setminus \{z_1, \dots, z_4\}$ (also surface, but ALG)

$M \setminus X^{\circ}$ = Hitchin section

3. Integrability

$$T^*\mathcal{F}_{\underline{r}^1} \times \cdots \times T^*\mathcal{F}_{\underline{r}^n} \times T^*\mathfrak{sl}(r,\mathbb{C})^g$$

$$\big\downarrow$$

$$\mathcal{X}^g_{\underline{r}^1,\dots,\underline{r}^n}(\alpha)$$

3. Integrability

$$\underbrace{T^*\mathcal{F}_{\underline{r}^1} \times \cdots \times T^*\mathcal{F}_{\underline{r}^n}}_{\text{Gelfand-Tsetlin}} \times T^*\mathfrak{sl}(r,\mathbb{C})^g$$

$$\longrightarrow \quad X^g_{\underline{r}^1,\dots,\underline{r}^n}(\alpha)$$

3. Integrability

$$\underbrace{T^*\mathcal{F}_{\underline{r}^1} \times \cdots \times T^*\mathcal{F}_{\underline{r}^n}}_{\text{Gelfand-Tsetlin}} \times \underbrace{T^*\mathfrak{sl}(r,\mathbb{C})^g}_{\text{Lie-Poisson}}$$

$$\downarrow$$

$$\mathcal{X}^g_{\underline{r}^1,\dots,\underline{r}^n}(\alpha)$$

3. Integrability

$$\underbrace{T^*\mathcal{F}_{\underline{r}^1} \times \cdots \times T^*\mathcal{F}_{\underline{r}^n}}_{\text{Gelfand-Tsetlin}} \times \underbrace{T^*\mathfrak{sl}(r,\mathbb{C})^g}_{\text{Lie-Poisson}}$$

$$\Big\downarrow \ \text{Poisson morphism}$$

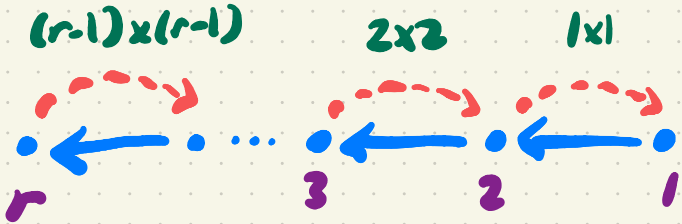
$$X^g_{\underline{r}^1,\dots,\underline{r}^n}(\alpha)$$

3. Integrability

$$\underbrace{T^*\mathcal{F}_{\underline{r}^1}\times\cdots\times T^*\mathcal{F}_{\underline{r}^n}}_{\text{Gelfand-Tsetlin}}\times\underbrace{T^*\mathfrak{sl}(r,\mathbb{C})^g}_{\text{Lie-Poisson}}$$

$$\xrightarrow{\text{Poisson morphism}}$$

$$\mathcal{X}^g_{\underline{r}^1,\dots,\underline{r}^n}(\alpha)$$



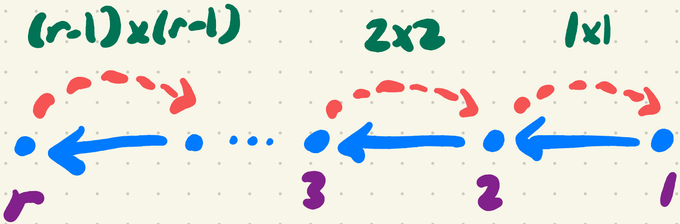
Assume c complete arms, $n-c$ minimal ones $(1,r)$

3. Integrability

$$\underbrace{T^*\mathcal{F}_{\underline{r}^1} \times \cdots \times T^*\mathcal{F}_{\underline{r}^n}}_{\text{Gelfand-Tsetlin}} \times \underbrace{T^*\mathfrak{sl}(r,\mathbb{C})^g}_{\text{Lie-Poisson}}$$

$$\Big\downarrow \ \text{Poisson morphism}$$

$$X^g_{\underline{r}^1,\dots,\underline{r}^n}(\alpha)$$



In complete arm, each $k \times k$ block contributes k invariants

$$c\,(1+2+\cdots+(r-1)) = \left[\frac{c\,(r-1)r}{2}\right]$$

natural invariants from the complete flags (Gelfand-Tsetlin Hamiltonians), $[(n-c)(\cdot\ \cdot)]$ f. in the minimal flags, and $[g(r^2-1)]$ from the loops $b_j \in sl(r,\mathbb{C})$

For each r, n, g, c, HK reduction by $SU(r)$ fixes $N(r,n,g,c)$ of the invariants and we show

$$\frac{c(r-1)r}{2} + (n-c)(r-1) + g(r^2-1) - N(r,n,g,c) = \frac{1}{2}\dim \mathcal{X}^{g}_{\underline{r}^1,\dots,\underline{r}^n}$$

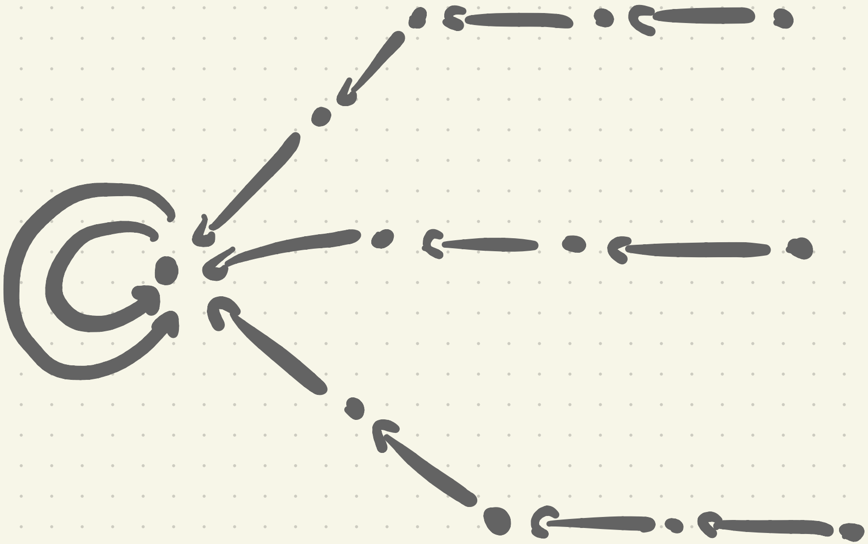
$\Rightarrow$ Existence of

sub-integrable systems

in meromorphic Hitchin systems

that do not see the complex

geometry of the algebraic curve



Questions?