Hyperkähler structures on holomorphic symplectic groupoids

Maxence Mayrand

UNIVERSITY OF TORONTO

June 7, 2020

Definition

A **hyperkähler manifold** is a Riemannian manifold (M, g) with three complex structures I, J, K that are Kähler with respect to g and satisfy the quaternionic identities $I^2 = J^2 = K^2 = IJK = -1$.

- e.g. $\mathbb{H}^n$, $T^*\mathbb{CP}^n$ (Calabi 1979)
- Kähler forms: ω_I, ω_J, ω_K
- (M, g, I, J, K) hyperkähler $\Longrightarrow (M, I, \omega_J + i\omega_K)$ holomorphic symplectic
- Partial converse by Yau's solution to the Calabi conjecture: (M, I, Ω) *compact* holomorphic symplectic and Kähler $\Longrightarrow$ hyperkähler

Upshot of the talk:

There is a canonical **hyperkähler** structure associated to any compact holomorphic **Poisson surface** (X, π) endowed with a Kähler metric.

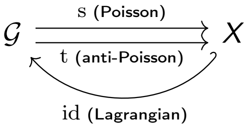
It lives on a local **symplectic groupoid** integrating (X, π).

It is constructed by lifting special **deformations** of the Poisson surface.

Holomorphic symplectic groupoids

(X, π) **holomorphic Poisson manifold**, $\pi \in H^0(\Lambda^2 \mathcal{T}_X)$, $[\pi, \pi] = 0$

- Induces a **Lie algebroid**: $\mathcal{T}_X^* \to X$
  Anchor: $\mathcal{T}_X^* \to \mathcal{T}_X$, $\alpha \mapsto i_\alpha \pi$
  Lie bracket: $[df, dg] = d\{f, g\}$
- Integrate $\mathcal{T}_X^*$ to a local Lie groupoid $\mathcal{G} \rightrightarrows X$
  $(\mathcal{G}, \Omega)$ holomorphic symplectic (induced by Ω_{can} on $\mathcal{T}_X^*$)



local symplectic groupoid:

- $\mathcal{G} \rightrightarrows X$ local Lie groupoid
- $(\mathcal{G}, \Omega)$ symplectic
- $\Gamma_{\mathrm{mult}} \subseteq \mathcal{G}^- \times \mathcal{G} \times \mathcal{G}$ is Lagrangian

- Conversely, given a local symplectic groupoid $\mathcal{G} \rightrightarrows X$, the symplectic form Ω descends to a Poisson structure π on X.

$$\{\text{Poisson manifolds}\} \longleftrightarrow \{\text{germs of local symplectic groupoids}\}$$

- Intriguing phenomenon: In many examples, $(\mathcal{G}, \Omega)$ carries a hyperkähler structure.

(X, π) holomorphic Poisson and Kähler

$(\mathcal{G}, \Omega) \rightrightarrows (X, \pi)$ local symplectic groupoid

(1) π non-degenerate, $\mathcal{G} = X \times X$ holomorphic symplectic $(\pi^{-1}, -\pi^{-1})$
$\mathcal{G}$ hyperkähler (if X is compact)
Proof. (Beauville 1983) Yau's solution to the Calabi conjecture + Bochner's principle

(2) $\pi = 0$, $\mathcal{G} = T^*X$
$\mathcal{G}$ hyperkähler on a neighborhood of $\mathrm{id} : X \xhookrightarrow{0} T^*X$
Proof. (Feix 1999, Kaledin 1999) Twistor theory

(3) $X = \mathfrak{g}^*$, $\mathfrak{g} = \mathrm{Lie}(G)$, G complex semisimple Lie group, $\mathcal{G} = T^*G$
$\mathcal{G}$ hyperkähler
Proof. (Kronheimer 1988) $T^*G \cong \{$gauge-theoretic moduli space$\}$ (infinite-dimensional hyperkähler quotient)

(4) **(this talk)** $\dim_{\mathbb{C}} X = 2$, X compact $\implies$ $\mathcal{G}$ hyperkähler

(X, I, π) holomorphic Poisson (compact Kähler)

$\pi : \mathcal{T}_X^* \to \mathcal{T}_X$ induces

$$\pi : H^1(X, \mathcal{T}_X^*) \longrightarrow H^1(X, \mathcal{T}_X) = \{\text{1st order deformations of } (X, I)\}$$

Theorem (Hitchin 2012)

Each class in the image of π integrates to a deformation (X, I_t), and π is deformed to holomorphic Poisson structures (X, I_t, π_t), $t \in \mathbb{C}$ small.

Symplectic foliation: Leaves stay unchanged.

Symplectic forms on leaves: Shifted by a global closed 2-form $\beta_t \in \Omega_X^2$
(L, Ω_L) symplectic leaf $\rightsquigarrow (L,\, \Omega_L + \iota_L^* \beta_t)$ (Gualtieri 2018)

Remark. β_t is a gauge transformation of Dirac structures: $e^{\beta_t} L_\pi = L_{\pi_t}$.

β_t determines the deformation completely. The condition that $\Omega_L + \iota_L^* \beta_t$ are holomorphic symplectic imposes non-linear algebraic constraints on β_t.

HK structures near a complex Lagrangian

(M, I, Ω) holomorphic symplectic (non-compact)
$X \subseteq M$ complex Lagrangian submanifold

Theorem (M.)

$$\{\textit{hyperkähler structure on a neighborhood of } X \textit{ in } (M, I, \Omega)\}$$
$$\updownarrow$$
$$\{\textit{deformation } (M, I_t, \Omega_t) \textit{ such that } \iota_X^* \Omega_t = t\omega,\ \omega \textit{ Kähler form on } X\}$$

Proof. Twistor theory. Idea: $X \times S^1 \subseteq M \times \mathbb{C}$ is totally real ($\mathbb{R}^n \subseteq \mathbb{C}^n$).
$X \times S^1 \to X \times S^1 : (x, \zeta) \mapsto (x, -\zeta)$ extends to holomorphic map
$M \times \mathbb{C}^* \to \overline{M \times \mathbb{C}^*}$. Glues $M \times \mathbb{C}$ and $\overline{M \times \mathbb{C}}$ to a twistor space
$Z \to \mathbb{CP}^1$. For all $x \in X$, $\zeta \mapsto (x, \zeta)$ is a real twistor line.

Corollary (Feix–Kaledin metrics)

$M = T^*X$, (X, ω) Kähler
$\Omega_t := \Omega_{can} + t\pi^*\omega$, where $\pi : T^*X \to X$
Ω_t is holomorphic symplectic (for a unique I_t) and $\iota_X^* \Omega_t = t\omega$
Hence, T^*X has a hyperkähler structure near the zero-section.

(X, π) holomorphic Poisson (compact Kähler)
$(\mathcal{G}, \Omega) \rightrightarrows (X, \pi)$ local symplectic groupoid, $\mathrm{id} : X \hookrightarrow \mathcal{G}$ Lagrangian

- **Goal.** Find a deformation $(\mathcal{G}, I_t, \Omega_t)$ such that $\mathrm{id}^*\Omega_t = t\omega$, ω Kähler
- **Hitchin's deformation of X in the direction of a Kähler form ω:**
  $\pi([\omega]) \in H^1(X, \mathcal{T}_X)$ tangent to a deformation (X, I_t, π_t)
  same symplectic leaves
  symplectic forms shifted by $\beta_t \in \Omega^2_X$
- **Lift the deformation to $\mathcal{G}$:** For small $t_1, t_2 \in \mathbb{C}$,
  $$\Omega_{t_1,t_2} := \Omega + \mathrm{s}^*\beta_{t_1} - \mathrm{t}^*\beta_{t_2}$$
  is a holomorphic symplectic form for a unique complex structure I_{t_1,t_2}.
  Remark. $(\mathcal{G}, \Omega_{t_1,t_2})$ is a Morita equivalence between (X, π_{t_1}) and (X, π_{t_2})
- $\mathrm{id}^*\Omega_{t_1,t_2} = \beta_{t_1} - \beta_{t_2}$
  $\mathrm{id}^*\Omega_{t,-t} = \beta_t - \beta_{-t} = 2(\omega t + \beta_3 t^3 + \beta_5 t^5 + \beta_7 t^7 + \cdots)$
- **Conclusion.** We need a Kähler form ω on X whose Hitchin deformation can be obtained by a family β_t such that $\beta_3 = \beta_5 = \beta_7 = \cdots = 0$.
- Can be done when $\dim_{\mathbb{C}} X = 2$ (for any ω)

To get the Hitchin deformation (X, I_t, π_t) from

$$\beta_t = \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \cdots$$

we need to solve the recursive equations

$$\beta_1 = \omega, \qquad d\beta_n = 0, \qquad \beta_n^{0,2} = \sum_{i+j=n} \beta_i \pi \beta_j .$$

β_n not unique: $\beta_n + \partial\bar{\partial} f_n$, $f_n \in C^\infty(X)$

Proposition (Key ingredient special to dimension 2)

On a Kähler surface (X, I, ω),

$$D : C^\infty(X) \longrightarrow C^\infty(X), \quad D(f) = *(i\partial\bar{\partial} f \wedge \omega)$$

is elliptic and self-adjoint. Hence, the Poisson equation $D(f) = g$ has a solution if and only if $\int_X g = 0$.

Can be used to pick f_{2n} such that $\beta_{2n+1} = 0$.

Theorem (M.)

Let (X, π) be a compact holomorphic Poisson surface endowed with a real-analytic Kähler form ω.

Let $(\mathcal{G}, I, \Omega) \rightrightarrows X$ be a local symplectic groupoid integrating (X, π).

Then, there is a unique hyperkähler structure (g, I, J, K) on a neighborhood of X in $\mathcal{G}$ such that $\mathrm{id}^*\omega_I = \omega$ and $\omega_J + i\omega_K = \Omega$.

Further questions

- Higher dimensions? Yau's theorem?
- For most X, $\mathcal{G}$ extends to an actual groupoid (Weinstein groupoid). When does (g, I, J, K) extend to the whole $\mathcal{G}$? When is it complete?
  - $(\mathbb{CP}^2, \sigma = 0, \omega_{FS})$ gives $\mathcal{G} = T^*\mathbb{CP}^2$ with Calabi metric (complete)
  - Non-zero Poisson structures on $\mathbb{CP}^2$ are specified by a **cubic curve**.
    Corollary. *There is a canonical hyperkähler metric associated to any cubic curve in $\mathbb{CP}^2$.*
    Are they complete?

Thank you