Quiver gauge theories and symplectic singularities

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Introduction

- Investigate properties of Coulomb branches of $3d$ $\mathcal{N} = 4$ quiver gauge theories
- Use mathematical construction of Braverman, Finkelberg and Nakajima
- Viewpoint is algebraic geometry over $\mathbb{C}$
- Plan:
  1. Background
  2. Coulomb branches, properties and examples
  3. Discuss proof that they have symplectic singularities

Background

Symplectic singularities

- Very interesting algebraic varieties with algebraic Poisson structures, generically (holomorphic) symplectic
- Movating examples:
    - Nilpotent cone of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, e.g.

$$\mathcal{N}_{\mathfrak{sl}_n} = \{A \in M_{n\times n}(\mathbb{C}) : \det(t - A) = t^n\}$$

    - Normalizations of nilpotent orbit closures
    - Kleinian singularities, e.g. $\mathbb{C}^2 /\!\!/ (\mathbb{Z}/n\mathbb{Z})$
- Interesting "representation theory" and enumerative geometry

Symplectic singularities

- Frequently arise in pairs, as Coulomb and Higgs branches of $3d$ $\mathcal{N} = 4$ gauge theories
- Subject* of symplectic duality program proposed by Braden-Licata-Proudfoot-Webster
- $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^\vee}$ are dual, where $\mathfrak{g}, \mathfrak{g}^\vee$ are Langlands dual Lie algebras

  (Here the "representation theory" is of $\mathfrak{g}$ and $\mathfrak{g}^\vee$, and more precisely of categories $\mathcal{O}$)

Symplectic singularities

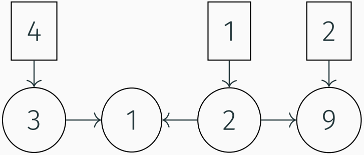
- A normal affine variety $X/\mathbb{C}$ has symplectic singularities if:
  1. Have a given symplectic form ω on smooth locus X^{reg}
  2. For some (any) resolution $\pi : Y \longrightarrow X$ of singularities, $\pi^*\omega$ extends to a regular 2–form on Y
- Coordinate ring $\mathbb{C}[X]$ gets Poisson bracket $\{\cdot, \cdot\}$
- Implies X has finitely many holomorphic symplectic leaves (Kaledin), and rational Gorenstein singularities (Beauville, Namikawa)

$$\mathcal{N}_{\mathfrak{sl}_n} = \bigsqcup_{\lambda \vdash n} \mathbb{O}_\lambda, \qquad \mathbb{O}_\lambda = \text{nilp. orbit of type } \lambda$$

Coulomb branches

Quiver gauge theories

- Associated to a quiver Q plus dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I_{\geq 0}$, where I is the set of vertices
- For example: the A_4 quiver $1 \to 2 \leftarrow 3 \to 4$, with $\mathbf{v} = (3, 1, 2, 9)$ and $\mathbf{w} = (4, 0, 1, 2)$



- To this data, physicists associate a $3d$ $\mathcal{N} = 4$ gauge theory. Its Higgs branch is the Nakajima quiver variety $\mathcal{M}_H(\mathbf{v}, \mathbf{w})$

The Coulomb branch

- Braverman-Finkelberg-Nakajima have given a construction of the Coulomb branch $\mathcal{M}_C(\mathsf{v}, \mathsf{w})$
- Let $D = \operatorname{Spec} \mathbb{C}[[t]]$. Define moduli space $\mathcal{R}$, of data
  1. Vector bundle $\mathcal{E}_i$ over D of rank v_i, for all $i \in I$
  2. Trivialization φ_i of $\mathcal{E}_i$ on $D^\times$, for all $i \in I$
  3. For all $i \in I$ and edges $i \to j$,

     $$s_i \in \operatorname{Hom}(\mathcal{O}_D \otimes_{\mathbb{C}} W_i, \mathcal{E}_i), \quad s_{i \to j} \in \operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_j)$$

     which remain regular under φ_i
- Action of $\mathsf{G}[[t]] = \prod_i \mathsf{GL}(\mathsf{v}_i)[[t]]$changing trivialization

The Coulomb branch

- BFN define Coulomb branch as affine scheme/$\mathbb{C}$

$$\mathcal{M}_C(\mathsf{v}, \mathsf{w}) = \operatorname{Spec} H_*^{\mathsf{G}[[t]]}(\mathcal{R})$$

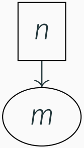
  Right side carries "convolution product", making it a commutative algebra

- BFN show $\mathcal{M}_C(\mathsf{v}, \mathsf{w})$ is irreducible normal affine variety, actually defined over $\mathbb{Z}$

- Also show $\mathcal{M}_C$ has a Poisson structure, symplectic on $\mathcal{M}_C^{reg}$

Type A_1

- Consider A_1 quiver datum



- $\mathcal{M}_C(m,n)$ has description due to Kamnitzer:

$$\left\{ \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \in M_2(\mathbb{C}[z]) \;:\; \begin{array}{rl} (i) & A \text{ monic, degree } m, \\ (ii) & \text{degrees } B, C < m, \\ (iii) & AD - BC = z^n \end{array} \right\}$$

Finite ADE type and affine type A

Theorem (Braverman-Finkelberg-Nakajima)

Suppose Q is oriented finite ADE, and let G_Q be the associated algebraic group (of adjoint type). Then

$$\mathcal{M}_C(\mathsf{v},\mathsf{w}) \cong \overline{\mathcal{W}}^{\lambda}_{\mu}$$

is a generalized affine Grassmannian slice for G_Q, where

$$\lambda = \sum_i \mathsf{w}_i \varpi_i^{\vee}, \qquad \lambda - \mu = \sum_i \mathsf{v}_i \alpha_i^{\vee}$$

are cocharacters of G_Q

Theorem (Nakajima-Takayama)

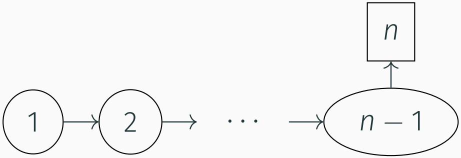
If Q is oriented affine type A, then $\mathcal{M}_C(\mathsf{v},\mathsf{w})$ is a Cherkis bow variety.

Finite ADE type

- For type A and μ dominant, then

$$\mathcal{M}_C(\mathsf{v},\mathsf{w}) \cong \overline{\mathbb{O}_\lambda} \cap \mathbb{S}_\mu$$

where $\mathbb{O}_\lambda, \mathbb{S}_\mu \subset \mathfrak{gl}_N$ nilpotent orbit/Slodowy slice, and $\lambda, \mu \vdash N$ partitions.



gives $\mathcal{M}_C(\mathsf{v},\mathsf{w}) \cong \mathcal{N}_{\mathfrak{sl}_n}$

- In finite ADE and affine A types, know decomposition of $\mathcal{M}_C(\mathsf{v},\mathsf{w})$ into symplectic leaves (finite ADE by Muthiah-W. and Kamnitzer-Webster-W.-Yacobi, affine type A Nakajima-Takayama)

General quivers

- Quivers without loops/multiple edges correspond to simply-laced Kac-Moody types

  Can *define*

$$\mathcal{M}_C(\mathsf{v}, \mathsf{w}) =: \begin{array}{c}\text{(generalized) affine Grassmannian slice} \\ \text{for Kac-Moody group } G_Q\end{array}$$

- Upshot: affine Grassmannian for G_Q is not defined in general

- BFN conjecture a version of the geometric Satake correspondence using $\mathcal{M}_C(\mathsf{v}, \mathsf{w})$

Symplectic singularities

Main result

Theorem (W.)

Let Q be a quiver without loops or multiple edges, and $\mathbf{v}, \mathbf{w}$ be arbitrary. Then $\mathcal{M}_C(\mathbf{v}, \mathbf{w})$ has symplectic singularities.

- This is conjectured by BFN for all Coulomb branches, not just quiver gauge theories
- Known already for dominant finite ADE type by Kamnitzer-Webster-W.-Yacobi, and affine type A by Nakajima-Takayama

Corollary

$\mathcal{M}_C(\mathbf{v}, \mathbf{w})$ has finitely many holomorphic symplectic leaves, and rational Gorenstein singularities.

First ingredient: partial resolutions

- Coulomb branches admit partial resolutions

$$\mathcal{M}_C^{\varkappa}(\mathsf{v},\mathsf{w}) \twoheadrightarrow \mathcal{M}_C(\mathsf{v},\mathsf{w})$$

 $\varkappa$ is cocharacter of certain “flavour symmetry” group

 Special case: Springer resolution $T^*Fl_n \to \mathcal{N}_{\mathfrak{sl}_n}$

- There is a completely integrable system

$$\varpi : \mathcal{M}_C(\mathsf{v},\mathsf{w}) \longrightarrow \mathfrak{t}/\!\!/W \cong \mathbb{C}^{\sum_i \mathsf{v}_i}$$

 It is faithfully flat, and comes from $H^*_{\mathsf{G}}(pt) \hookrightarrow H_*^{\mathsf{G}[[z]]}(\mathcal{R})$.

 Special case: Gelfand-Tsetlin integrable system

$$\mathcal{N}_{\mathfrak{sl}_n} \to \mathbb{C}^{\frac{n(n-1)}{2}}, \quad A \longmapsto \prod_{i=1}^{n-1} \big(\text{coefficients of } \det(t - A_i)\big)$$

Final ingredient: open subsets

Using results of Beauville and Bellamy-Schedler, sufficient to give open subsets

$$\begin{array}{ccccc} \mathcal{M}_C^{\varkappa}(\mathsf{v},\mathsf{w}) & \longrightarrow & \mathcal{M}_C(\mathsf{v},\mathsf{w}) & \longrightarrow & \mathfrak{t}/\!\!/W \\ \uparrow & & & & \uparrow \\ U & & \longrightarrow & & V \end{array}$$

so that

(i) diagram is Cartesian

(ii) $\mathrm{codim}_{\mathbb{C}} V = 4$,

(iii) U is smooth and symplectic

Then $\mathrm{codim}_{\mathbb{C}}(\mathcal{M}_C^{\varkappa}(\mathsf{v},\mathsf{w}))^{sing} \geq 4$

Second ingredient: integrable system

Theorem (W.)

1. Étale neighbourhood of any fiber of ϖ is isomorphic to a product

$$\mathcal{M}_C^{\varkappa}(\mathsf{v}^{(1)}, \mathsf{w}^{(1)}) \times \cdots \times \mathcal{M}_C^{\varkappa}(\mathsf{v}^{(\ell)}, \mathsf{w}^{(\ell)})$$

2. For generic $\varkappa$, can choose V such that over V these products are smooth and symplectic.

   Establishes diagram on previous page, so proves theorem.

Questions

- Enumerate symplectic leaves, and their transverse slices?
- Is $\mathcal{M}_C^{\varkappa} \twoheadrightarrow \mathcal{M}_C$ a $\mathbb{Q}$–factorial terminalization, for generic $\varkappa$? When is it a resolution?
- Quivers with loops and/or multiple edges? Symmetrizable types?
- Other Coulomb branches?

Thank you for listening!

I refuse to answer that question on the grounds that I don't know the answer.

- Douglas Adams