BFN Springer Theory and the Hikita conjecture

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Higgs and Coulomb branches

G a complex reductive group, V a representation of G
Physicists define a gauge theory from G, V and two spaces:

- Higgs branch

$$\mathcal{M}_H(G,V) := T^*V /\!\!/\!\!/\!\!/_\chi\, G = \mu^{-1}(0) /\!\!/_\chi\, G$$

- Coulomb branch

$$\text{roughly} \quad \mathcal{M}_C(G,V) := \operatorname{Spec} H_*(Maps(\mathbb{P}^1, V/G))$$

Example

$G = \mathbb{C}^\times, V = \mathbb{C}^n$

- Higgs branch

$$\mathcal{M}_H(G,V) = \{A \in M_n(\mathbb{C}) : A^2 = 0, \operatorname{rank} A \leq 1\}$$

- Coulomb branch

$$\mathcal{M}_C(G,V) = \mathbb{C}^2 /\!\!/ \mathbb{Z}/n$$

Coulomb branch algebra

Definition due to Braverman-Finkelberg-Nakajima

- $\mathcal{K} = \mathbb{C}((t)),\ \mathcal{O} = \mathbb{C}[[t]],\ Gr_G := G(\mathcal{K})/G(\mathcal{O})$
- $T_{G,V} := \{([g], v) \in Gr_G \times V \otimes \mathcal{K} : v \in g(V \otimes \mathcal{O})\} \to Gr_G$, a vector bundle with fibre $V \otimes \mathcal{O}$
- $Z_{G,V} := T_{G,V} \times_{V \otimes \mathcal{K}} T_{G,V}$, Steinberg variety
- $\mathcal{A}_{G,V} = H_*^{G(\mathcal{K})}(Z_{G,V})$, commutative convolution algebra
- $\mathcal{A}_{G,V}^{\hbar} = H_*^{G(\mathcal{K}) \rtimes \mathbb{C}^\times}(Z_{G,V})$, non-commutative deformation

Definition

1. $\operatorname{Spec} \mathcal{A}(G, V)$ is the **Coulomb branch**
2. $\mathcal{A}_{G,V}^{\hbar}$ is the **Coulomb branch algebra**

Hikita Conjecture

There are many (conjectural) relationships between the Higgs and Coulomb branches; today, we examine the Hikita conjecture.

Since Gr_G has connected components labelled by $\pi_1(G)$:

$$\mathcal{A}^{\hbar}_{G,V} = \bigoplus_{\sigma \in \pi_1(G)} \mathcal{A}^{\hbar}_{G,V}(\sigma)$$

We have $\chi : G \to \mathbb{C}^\times$ and so $\pi_1(G) \to \mathbb{Z}$, so

$$\mathcal{A}^{\hbar}_{G,V} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^{\hbar}_{G,V}(n)$$

$$B(\mathcal{A}^{\hbar}_{G,V}) = \mathcal{A}^{\hbar}_{G.V}(0) \Big/ \mathbb{C}\{ab : a \in A^{\hbar}_{G,V}(-n), b \in A^{\hbar}_{G,V}(n), n > 0\}$$

Let $M = \oplus_{k \in \mathbb{Z}} M(k)$ be a $\mathcal{A}^{\hbar}_{G,V}$-module with $M(k) = 0$ for $k > n_0$.
$B(\mathcal{A}^{\hbar}_{G,V})$ acts on $M(n_0)$.
If $M(n_0) = \mathbb{C}[\hbar]$, $B(\mathcal{A}^{\hbar}_{G,V}) \to \mathbb{C}[\hbar]$, the **highest weight** of M.

Hikita and highest weights

We have a Kirwan map

$$H^*_{G\times\mathbb{C}^\times}(pt) \to H^*_{\mathbb{C}^\times}(\mathcal{M}_H(G,V))$$

And the Gelfand-Tsetlin subalgebra, a complete integrable system

$$H^*_{G\times\mathbb{C}^\times}(pt) \to \mathcal{A}^\hbar_{G.V}(0)$$

Conjecture

$H^*_{\mathbb{C}^\times}(\mathcal{M}_H(G,V)) \cong B(\mathcal{A}^\hbar_{G,V})$ as $H^*_{G\times\mathbb{C}^\times}(pt)$-algebras.

$$\operatorname{Spec} H^*_{\mathbb{C}^\times}(\mathcal{M}_H(G,V)) = \operatorname{Spec} B(\mathcal{A}^\hbar_{G,V}) \subset \operatorname{Spec} H^*_{G\times\mathbb{C}^\times}(pt)$$

Thus Hikita conjecture predicts a bijection

$$\pi_0(\mathcal{M}_H(G,V)^{\mathbb{C}^\times}) = \{ \text{ highest weights for } \mathcal{A}^\hbar_{G,V}\text{-modules } \}$$

$$v \mapsto \left(H^*_{G\times\mathbb{C}^\times}(pt) \to H^*_{\mathbb{C}^\times}(\mathcal{M}_H(G,V)) \to H^*_{\mathbb{C}^\times}(\{v\}) = \mathbb{C}[\hbar]\right)$$

Modules from BFN Springer fibres

Goal: construct modules for Coulomb branch algebras. The **BFN Springer fibre** is the fibre of $T_{G,V}$ over $v \in V \otimes \mathcal{K}$.

$$F_v = \{[g] \in Gr_G : v \in g(V \otimes \mathcal{O})\}$$

Let L_v be the stablizer of v in $G(\mathcal{K}) \rtimes \mathbb{C}^\times$. L_v acts on F_v.

Theorem

There is an action of $\mathcal{A}^{\hbar}_{G,V}$ on $H^{L_v}_*(F_v)$.

Example

$$G = GL_n, \;\; V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m), \;\; v \in V \text{ injective}$$
$$F_v = \{L \subseteq \mathcal{O}^n\}, \;\text{positive part of the affine Grassmannian}$$
$$= \sqcup_{k \in \mathbb{N}} \overline{Gr^{k\omega_1}} = \sqcup \{L \subseteq \mathcal{O}^n : \dim \mathcal{O}^n / L = k\}$$

Modules from fixed points

Theorem

Assume that $v \in V$ be χ-stable, s.t. $[v] \in V /\!\!/_\chi G$ is $\mathbb{C}^\times$-fixed.

- $F_v(n) = \emptyset$ *if* $n > 0$ *and* $F_v(0) = pt$.
- $F_v(n)$ *is a finite-dimensional projective variety, if* $n < 0$.
- $H_*^{\mathbb{C}^\times}(F_v)$ *is a highest weight module with highest weight given by* $H^*_{G \times \mathbb{C}^\times}(pt) \to H^*_{\mathbb{C}^\times}(V /\!\!/_\chi G) \to H^*_{\mathbb{C}^\times}(pt)$.

This establishes the Hikita bijection for those fixed points which live in the Kahler quotient $(V /\!\!/_\chi G)^{\mathbb{C}^\times} \subset \mathcal{M}_H(G, V)^{\mathbb{C}^\times}$
If we choose a different G-invaraint Lagrangian $L \subset V \oplus V^*$, then $\mathcal{M}_C(G, L) \cong \mathcal{M}_C(G, V)$.
We establish the Hikita bijection for all fixed points in some $L /\!\!/_\chi G \subset \mathcal{M}_H(G, V)$.