Semitoric systems with multi-pinched fibers

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Workshop on Lie Theory and Integrable Systems in Symplectic and Poisson Geometry
June 7, 2020

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2 Toric systems
- Definition
- Classification

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- Invariants

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Section 1

Integrable systems

Integrable systems

Definition

An integrable system consists of

- a symplectic symplectic manifold (M^{2n}, ω);
- a Hamiltonian $\mathfrak{t}^n$-action $\rho\colon \mathfrak{t}^n \to \mathfrak{X}(M)$;
- a momentum map $\mu\colon M \to (\mathfrak{t}^n)^* \simeq \mathbb{R}^n$;
- so that the critical points of μ forms a null set.

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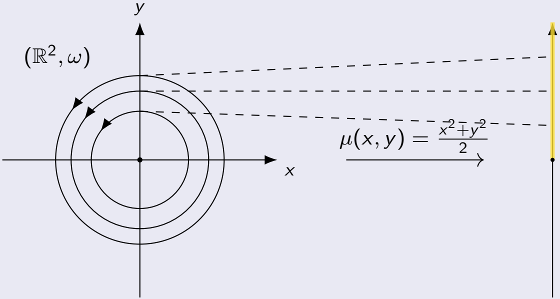
For any integrable system (M, ω, ρ, μ):

- $X_{\langle \mu, a\rangle} = \rho(a)$ is parallel to fibers of μ for $a \in \mathfrak{t}^n$;
- $\rho(a)$ and $\rho(b)$ commute for $a, b \in \mathfrak{t}^n$;
- any regular fiber of μ is Lagrangian.

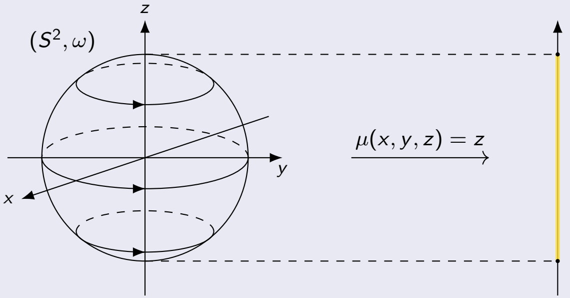
Example 1

Harmonic oscillator



Example 2

2-sphere



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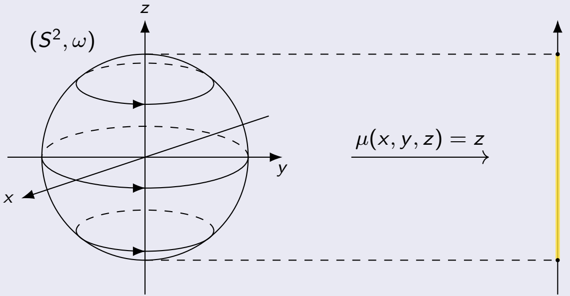

Remark

The Hamiltonian vector field act as a circle action.

Section 2

Toric systems

Toric systems

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In the examples of the harmonic oscillator and the 2-sphere, the Hamiltonian vector fields are periodic, or generates an S^1-action. Those are 2D examples of toric integrable systems where we have torus actions.

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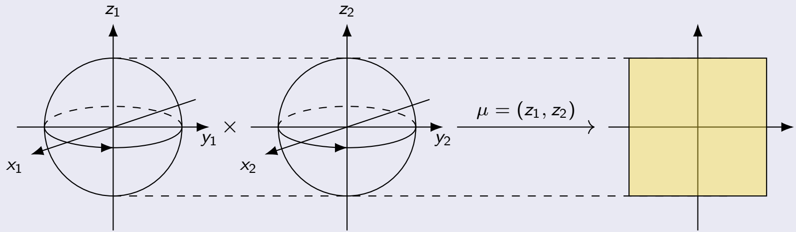
Definition

An integrable system (M, ω, ρ, μ) is toric if ρ integrates to a Lie group action $\tilde{\rho}\colon T^n \to \mathrm{Ham}(M, \omega)$.

Example 3

$(S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2})$



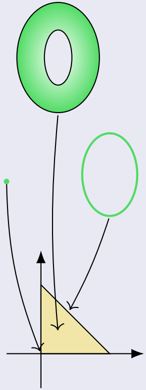
Example 4

$(\mathbb{C}P^2, \omega_{\mathrm{FS}})$

$\mathbb{C}P^2 = \{[z_0 : z_1 : z_2]\}$

$\mu([z_0 : z_1 : z_2]) = \left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2} \right)$

$(\mathbb{C}P^2, \omega_{\mathrm{FS}}) \xrightarrow{\mu}$



Toric systems

Isomorphisms of toric systems

A toric system $(M_1, \omega_1, \rho_1, \mu_1)$ is isomorphic to $(M_2, \omega_2, \rho_2, \mu_2)$ if

$$\begin{tikzcd} (M_1, \omega_1) \arrow[r, "\varphi"] \arrow[d, "\mu_1"'] & (M_2, \omega_2) \arrow[d, "\mu_2"] \\ \mathbb{R}^2 \arrow[r, "G"] & \mathbb{R}^2 \end{tikzcd}$$

where φ is a symplectomorphism and G is a diffeomorphism.

Toric systems: classification

Theorem (Atiyah, Guillemin–Sternberg, Delzant 1980s)

$$
\begin{array}{ccc}
\big\{\textit{compact toric systems}\big\} \Big/ \textit{isomorphisms} & \longrightarrow & \big\{\textit{Delzant polytopes}\big\} \Big/ AGL(n, \mathbb{R}) \\
[(M, \omega, \rho, \mu)] & \longmapsto & [\mu(M)]
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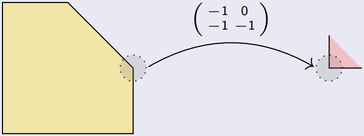
Delzant polytopes



Delzant polytope:
every corner is locally
the standard corner up to
the action of $AGL(n,\mathbb{Z})$.

Section 3

Semitoric systems

Generalized coupled angular momentum

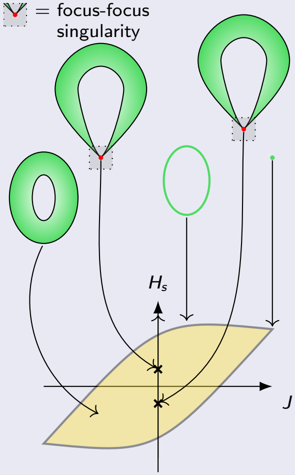
Examples: Hohloch–Palmer 2018

Let $M = S^2 \times S^2$ with Cartisian coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ and $\omega = R_1 \omega_{S^2} \oplus R_2 \omega_{S^2}$ where $0 < R_1 < R_2$. Consider integrable systems $(M, \omega, \rho_s, \mu_s = (J, H_s))$ with a parameter $s \in [0, 1]$ where

$$J = R_1 z_1 + R_2 z_2,$$
$$H_s = (1-s)^2 z_1 + s^2 z_2 + 2s(1-s)(x_1 y_1 + x_2 y_2).$$

The systems behave differently as R_1, R_2, and s varies.

Example 5

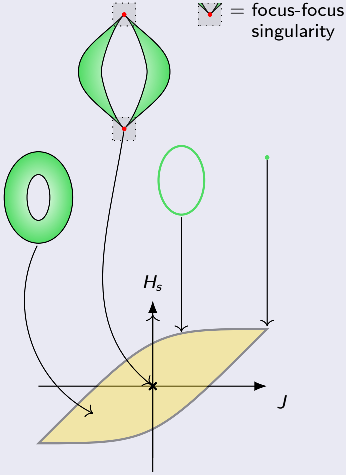


$$R_1 = R_2 = 1,$$

$$s - \frac{1}{2} \text{ is small positive}$$

$$\left(S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2}\right) \xrightarrow{\mu_s = (J, H_s)} \mathbb{R}^2$$

Example 6



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Semitoric vs toric

Comparing with toric systems

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This is an example of a semitoric system.

Semitoric systems

Definition

A 4D integrable system $(M^4, \omega, \rho, \mu = (J, H))$ is semitoric if ρ integrates to a Lie group action $\tilde{\rho} \colon S^1 \times \mathbb{R} \to \mathrm{Ham}(M, \omega)$, J is proper, and all singularities are of either elliptic or focus-focus type.

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Theorem (Eliasson 1984)

For an integrable system (M, ω, ρ, μ) in a neighborhood of any nondenerate singular point of μ, there are symplectic coordinates x_i, y_i and $q = (q_1, \dots, q_n)\colon \mathbb{R}^{2n} \to \mathbb{R}^n$ where q_i can be

- *regular:* $q_i = y_i$;
- *elliptic:* $q_i = \frac{1}{2}(x_i^2 + y_i^2)$;
- *hyperbolic:* $q_i = x_i y_i$;
- *focus-focus:* $q_{i-1} = x_{i-1} y_i - x_i y_{i-1}$ *and* $q_i = x_{i-1} y_{i-1} + x_i y_i$;

such that q_i Poisson commutes with components of μ.

Semitoric systems

In the absense of hyperbolic components Eliasson's theorem implies that q and μ are related by a diffeomorphism of the codomain, which means that $(\mathbb{R}^{2n}, \omega_0, \rho_0, q)$ is an local model of (M, ω, ρ, μ). Thus in the definition of semitoric systems we only rule out degenerate and hyperbolic singularities.

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Isomorphisms of semitoric systems

Two semitoric systems $(M_i, \omega_i, \rho_i, \mu_i = (J_i, H_i))$, $i = 1, 2$ are isomorphic if

$$\begin{tikzcd} (M_1, \omega_1) \arrow[r, "\varphi"] \arrow[d, "\mu_1"'] & (M_2, \omega_2) \arrow[d, "\mu_2"] \\ \mathbb{R}^2 \arrow[r, "G"] & \mathbb{R}^2 \end{tikzcd}$$

commutes where φ is a symplectomorphism and $G(J_2, H_2) = (J_1, f(J_1, H_1))$ for some smooth function f with $\frac{\partial f}{\partial H_1} > 0$.

Semitoric systems: classification

Theorem (Palmer–Pelayo–T 2019)

$$\left\{\textit{semitoric systems}\right\} \Big/ \textit{isomorphisms} \longrightarrow \left\{\begin{matrix}\textit{marked VPIA Delzant}\\ \textit{polytopes with labels}\end{matrix}\right\} \Big/ \textit{VPIA group}$$

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A marked VPIA Delzant polytope with labels has three parts

- *a marked VPIA Delzant polytope;*
- *a focus-focus label for each focus-focus value;*
- *a twisting covector for each focus-focus value.*

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VPIA stands for vertically piecewise integral affine.

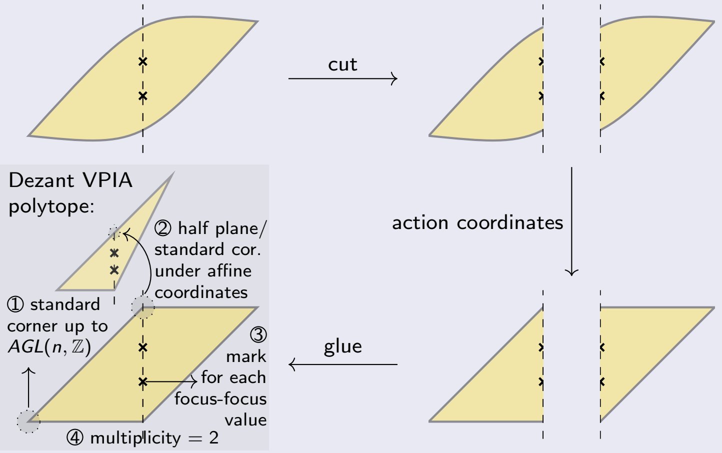
Invariants: marked polytope

The marked VPIA Delzant polytope

It is so called a Delzant VPIA polygon Δ which captures the global affine structure of $\mu(M)$.
We cut $\mu(M)$ by vertical lines through focus-focus values into pieces.
Each piece is turned into a polygon piece under action coordinates. Then we glue the polygon pieces to get a polygon.
Each mark represents a focus-focus value.

Invariants: marked polytope

Example 5



Invariants: marked polytope

Group action on polygons

As a polygon in $\mathbb{R}^2$, Δ is not unique.
Per different choices of the action coordinates, there are many different resulting polygons Δ, which are all related to the VPIA group $G_{\mathbf{j}}$ where $\mathbf{j}$ is the set of the abscissae of focus-focus values. The group $G_{\mathbf{j}}$ is generated by T, $\mathfrak{t}_j$ for $j \in \mathbf{j}$ and $\mathfrak{y}_b$ for $b \in \mathbb{R}$.

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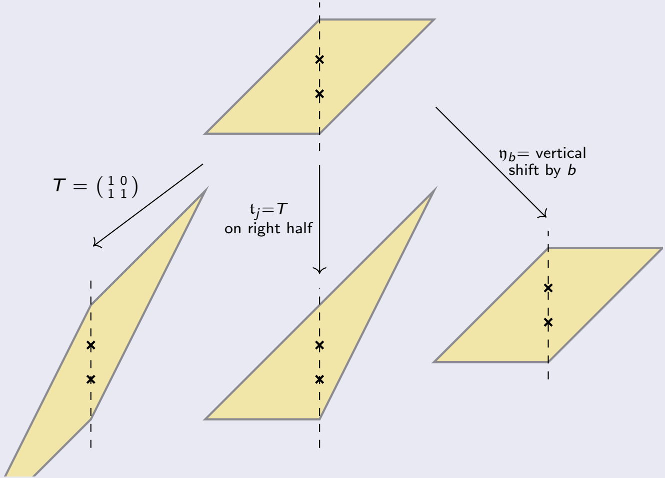

Invariants: focus-focus labels

The focus-focus labels

For every focus-focus value $\times$ with multiplicity k, we have a tuple

$$\mathsf{l} = (\mathrm{s}_0, \mathrm{g}_{0,1}, \ldots, \mathrm{g}_{0,k-1})$$

of k independent formal power series which captures the local affine structure near the focus-focus value. The tuple l is called the focus-focus label at $\times$.

Invariants: focus-focus labels

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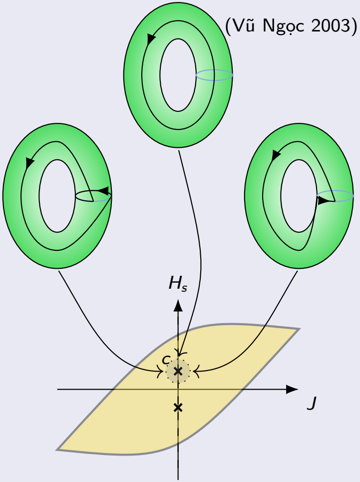

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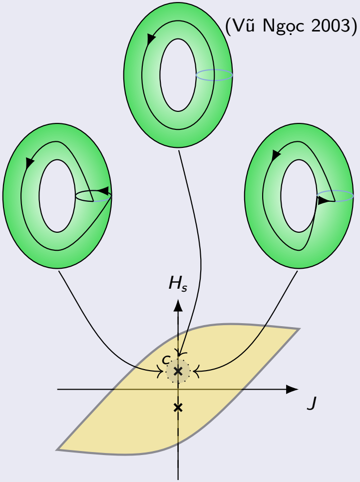

For any regular value c near the focus-focus value $\times$:

$\tau_1(c) \in S^1$ — travel time along X_J,

$\tau_2(c) > 0$ — travel time along X_H.

Functions τ_1 and τ_2 diverge at $\times$, and the monodromy causes τ_1 to increase by 2π every cycle c moves around $\times$.

Invariants: focus-focus labels

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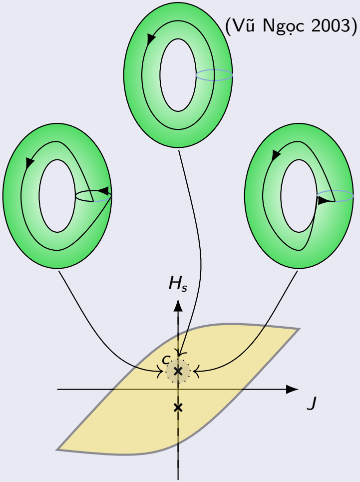

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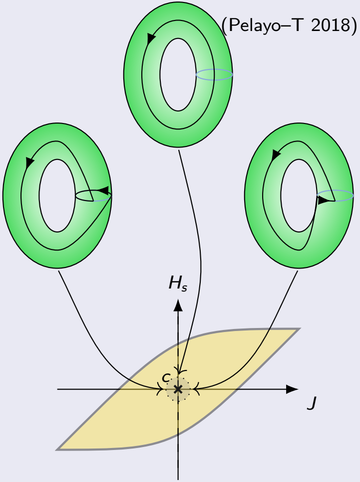
$\kappa = -\Im \ln c \, \mathrm{d}c_1 - \Re \ln c \, \mathrm{d}c_2$.
$\sigma = \tau_1 \, \mathrm{d}c_1 + \tau_2 \, \mathrm{d}c_2 - \kappa$.
Take function S_0 vanishing at $\times$ and $\mathrm{d}S_0 = \sigma$.
Then s_0 is the Taylor series of S_0 at origin.

Invariants: focus-focus labels

Example 6



Invariants: focus-focus labels

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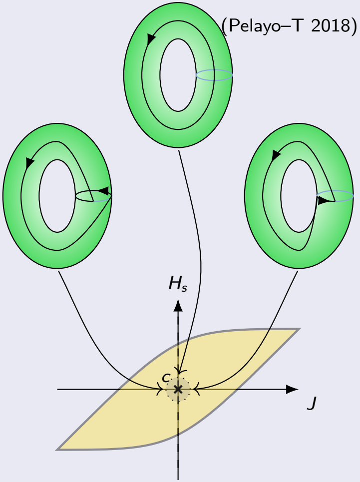

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Invariants: focus-focus labels

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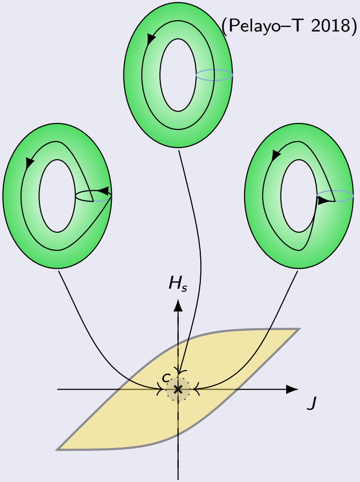

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Take function S_0 vanishing at $\times$ and $\mathrm{d}S_0 = \sigma$.
Then s_0 is the Taylor series of S_0 at origin.

Invariants: focus-focus labels

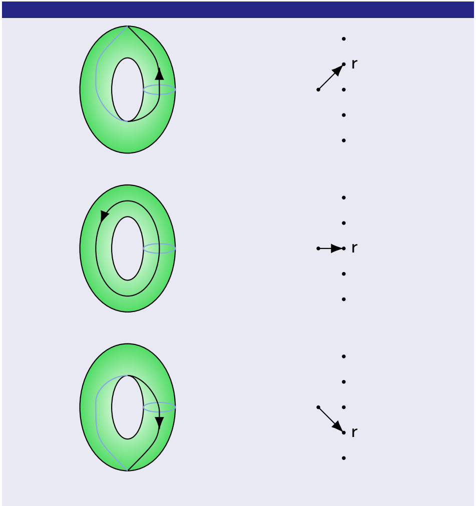
In Example 6 both focus-focus points lie in the same fiber. There is a choice which is m_0 and which is m_1. After then, the Eliasson's theorem gives symplectic coordinates $(x_1^a, y_1^a, x_2^a, y_2^a)$ near m_a, for $a = 0, 1$, and so that $q(x_1^a, y_1^a, x_2^a, y_2^a) = \mu(E^a(z))$, for $z \in M$ and E^a a diffeomorphism.
First, the series s_0 should be expanded under coordinates E^0.
Second, the series $\mathrm{g}_{0,1}$ is the Taylor series of the second component of $G_{0,1} = E^1 \circ (E^0)^{-1}$.

Invariants: twsting covector

The twsting covectors

For every focus-focus value $\times$ we construct a covector r in a prescibed cotangent space with as many possible values as $\mathbb{Z}$ which captures the Dehn twist of the trajectory of X_H in nearby fibers over $\times$.

Invariants: twsting covector



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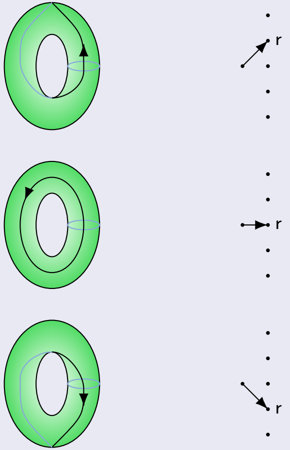

The twsting covector r takes value in the cotangent space of the plane at the focus-focus value defined in a certain way, and is noncanonically identified with integers.

Section 4

Historical review

Historical results

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A semitoric system $(M^4, \omega, \rho, \mu = (J, H))$ is simple if J is injective on singular points of F.

Historical results

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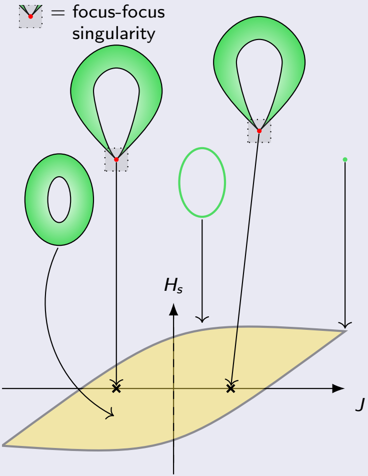
A semitoric system $(M^4, \omega, \rho, \mu = (J, H))$ is simple if J is injective on singular points of F.

Neither of Example 5 nor 6 is simple.

Example 7

simple semitoric system



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Simple semitoric systems: classification

Theorem (Pelayo–Vũ Ngọc 2007)

$$\left\{\textit{simple semitoric systems}\right\}\Big/\textit{isomorphisms} \quad \to \quad \left\{\textit{five invariants}\right\}\Big/\textit{a discrete group}$$

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Five invariants:

- *a polytope invariant;*
- *the number of focus-focus values;*
- *a height invariant* $\mathrm{h} > 0$ *for each focus-focus value;*
- *a Taylor series for each focus-focus value;*
- *an twisting index* $\mathrm{k} \in \mathbb{Z}$ *for each focus-focus value.*

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Five invariants: ——— *Palmer–Pelayo–T 2019 invariants*

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- *an twisting index* $\mathrm{k} \in \mathbb{Z}$ *for each focus-focus value.* } *twisting covector*

A review of sympletic classifications of integrable systems

	local	semi-local	global
toric			
simple semitoric			
semitoric			

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Thank you!