Reduction and Darboux-Moser-Weinstein theorems for symplectic Lie algebroids

Reyer Sjamaar

Department of Mathematics
Cornell University

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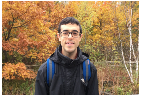
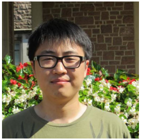
Co-authors

Yi Lin (Georgia Southern University)



Yiannis Loizides (Pennsylvania State University)



Yanli Song (Washington University)

Darboux-Moser-Weinstein for Lie algebroids

Theorem

Let $A \longrightarrow M$ be a Lie algebroid and let ω_0 and ω_1 be A-symplectic forms on M. Let $j\colon N \longrightarrow M$ be a submanifold which cleanly intersects A and such that A admits an Euler-like section along N. Suppose that $\omega_{0,x} = \omega_{1,x}$ for all $x \in N$. Then there exist open neighbourhoods U_0 and U_1 of N in M and a Lie algebroid isomorphism $\varphi\colon A|_{U_0} \xrightarrow{\cong} A|_{U_1}$ such that $\varphi^*\omega_1 = \omega_0$ and $\varphi|_{A_x} = \mathrm{id}_{A_x}$ for all $x \in N$.

Terms in red to be defined later.

Some recent prior work

- V. Guillemin, E. Miranda, and A. Pires, *Symplectic and Poisson geometry on b-manifolds*, Adv. Math. **264** (2014), 864–896.
- R. Klaasse, *Geometric structures and Lie algebroids*, Ph.D. thesis, Utrecht University, 2017, `arxiv.org:1712.09560`.

Also compare

- R. Fernandes and I. Mărcuţ, *Local models around Poisson submanifolds*, in preparation.

But: our submanifolds are not "locally split".

Marsden-Weinstein reduction for symplectic Lie algebroids

Let G be a Lie group.

Theorem

Let $(A \longrightarrow M, \omega, \rho, \mu)$ be a Hamiltonian G-Lie algebroid. Let $\mathcal{O}$ be a coadjoint orbit of G and let $\sigma_{\mathcal{O}}$ the Kirillov-Kostant-Souriau symplectic form on $\mathcal{O}$. Let $N = \mu^{-1}(\mathcal{O})$ and let $j \colon N \longrightarrow M$ be the inclusion. Suppose that the G-action on N is proper and free. Then the quotient manifold $Q = N/G$ is equipped with a quotient Lie algebroid C and a symplectic form $\omega_C \in \Omega^2_C(Q)$ such that $q_\sharp^* \omega_C = j_\sharp^* \omega - \mathbf{an}^* \mu_N^* \sigma_{\mathcal{O}}$.

Recent prior work

- H. Bursztyn and M. Crainic, *Dirac structures, momentum maps, and quasi-Poisson manifolds*, The breadth of symplectic and Poisson geometry, Festschrift in Honor of Alan Weinstein (J. Marsden and T. Ratiu, eds.), Progress in Mathematics, vol. 232, Birkhäuser Boston, Boston, MA, 2005, pp. 1–40.
- M. Gualtieri, S. Li, Á. Pelayo, and T. Ratiu, *The tropical momentum map: a classification of toric log symplectic manifolds*, Math. Ann. **367** (2017), no. 3-4, 1217–1258.

Guillemin-Sternberg normal form near zero fibre of moment map

Theorem

Let $(A \longrightarrow M, \omega, \rho, \mu)$ be a Hamiltonian G-Lie algebroid. Suppose that the G-action on the zero fibre $N = \mu^{-1}(0)$ is proper and free. Then N is a coisotropic submanifold of M and there is a nice local model for (A, ω, ρ, μ) in a G-invariant open neighbourhood of N.

Motivation

Goal: solve problem posed by

- V. Guillemin, E. Miranda, and J. Weitsman, *On geometric quantization of b-symplectic manifolds*, Adv. Math. **331** (2018), 941–951.

"Quantization commutes with reduction" for log symplectic, or *b*-symplectic manifolds. See Yiannis Loizides' presentation in this workshop.

DMW for Lie algebroids, again

Theorem

Let $A \longrightarrow M$ be a Lie algebroid and let ω_0 and ω_1 be A-symplectic forms on M. Let $j\colon N \longrightarrow M$ be a submanifold which cleanly intersects A and such that A admits an Euler-like section along N. Suppose that $\omega_{0,x} = \omega_{1,x}$ for all $x \in N$. Then there exist open neighbourhoods U_0 and U_1 of N in M and a Lie algebroid isomorphism $\varphi\colon A|_{U_0} \xrightarrow{\cong} A|_{U_1}$ such that $\varphi^*\omega_1 = \omega_0$ and $\varphi|_{A_x} = \mathrm{id}_{A_x}$ for all $x \in N$.

Symplectic Lie algebroids

Let A be a Lie algebroid over a manifold M with anchor map $\mathbf{an} = \mathbf{an}_A \colon A \to TM$, Lie bracket

$$[\cdot,\cdot] = [\cdot,\cdot]_A \colon \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A),$$

and de Rham-Eilenberg-Chevalley complex $(\Omega_A^\bullet(M), d_A)$. Elements of $\Omega_A^\bullet(M)$ are sections of the exterior algebra $\Lambda^\bullet(A^*)$, and are called *$\mathscr{LA}$-forms*, or *A-forms*, or just *forms*. The differential d_A is defined in terms of the anchor and the Lie bracket.

Definition (Nest-Tsygan, 2001)

An *A-symplectic form on M*, or a *symplectic form on A*, is an alternating bilinear form $\omega \in \Omega_A^2(M)$ that is d_A-closed, i.e. $d_A\omega = 0$, and non-degenerate. A *symplectic Lie algebroid* is a Lie algebroid equipped with a symplectic form.

Symplectic Lie algebroids are Poisson

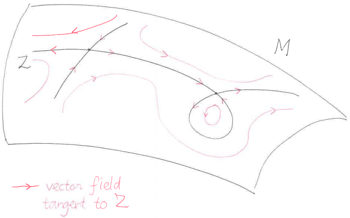
If $(A \to M, \omega)$ is a symplectic Lie algebroid, the map

$$\Pi\colon T^*M \xrightarrow{\mathbf{an}^*} A^* \xrightarrow{\omega^\flat} A \xrightarrow{\mathbf{an}} TM,$$

is the anchor of a Poisson structure on M, where $\mathbf{an}^*\colon T^*M \to A^*$ is the vector bundle map dual to the anchor of A.

Symplectic Lie algebroids: examples

- A constant rank Poisson structure Π defines an A-symplectic structure on the Lie algebroid $A = \mathrm{im}(\Pi) \hookrightarrow TM$.
- A *normal crossing divisor* of M is a subset Z which in suitable coordinates looks like a union of coordinate hyperplanes in $\mathbf{R}^n$. (C^∞ analogue of nc divisor in algebraic/analytic geometry.)



The vector fields tangent to (every component of) Z are the sections of a Lie algebroid, the *log tangent bundle* $TM(-\log Z)$ of (M, Z). A symplectic structure on this Lie algebroid is a *log symplectic structure* on (M, Z). If Z is a single smooth hypersurface, we call a log symplectic structure also a *b-symplectic structure*.

Some constant coefficient log symplectic forms on $\mathbf{R}^4$

Example (four hyperplanes)

Let $M = \mathbf{R}^4$, $Z = \{ x_1 x_2 x_3 x_4 = 0 \}$,

$$\omega = \sum_{1 \le i < j \le 4} a_{ij}\, d\log x_i\, d\log x_j = \sum_{1 \le i < j \le 4} a_{ij} \frac{dx_i}{x_i} \frac{dx_j}{x_j}$$

with (a_{ij}) invertible antisymmetric. Not all are linearly equivalent.

Example (three hyperplanes)

Let $M = \mathbf{R}^4$, $Z = \{ x_1 x_2 x_3 = 0 \}$,

$$\omega = b_{12} \frac{dx_1}{x_1} \frac{dx_2}{x_2} + b_{13} \frac{dx_1}{x_1} \frac{dx_3}{x_3} + b_{23} \frac{dx_2}{x_2} \frac{dx_3}{x_3} + \frac{dx_3}{x_3}\, dx_4$$

with $b_{12} \neq 0$. Not all are linearly equivalent.

Cleanly intersecting a Lie algebroid

Definition

Let $A \to M$ be a Lie algebroid and $j\colon N \to M$ a smooth map (e.g. inclusion of a submanifold). Then j *cleanly* (resp. *transversely*) *intersects* A if the tangent map $Tj\colon TN \to TM$ cleanly (resp. transversely) intersects the anchor $\mathbf{an}\colon A \to TM$.

Fact

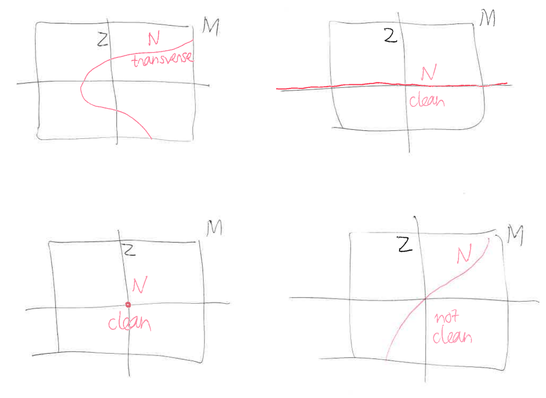
j cleanly intersects A if and only if the pullback $j^! A = TN \times_{TM} A$ is a well-defined Lie algebroid over N.

Fact

j cleanly intersects A if and only if the subspace $\operatorname{im}(T_x j) + \operatorname{im}(\mathbf{an}_{j(x)})$ of $T_{j(x)}M$ has dimension independent of $x \in N$. j is transverse to A if and only if j is transverse to all orbits of A.

Example

Let $M = \mathbf{R}^2$, $A = TM(-\log Z)$, where $Z = \{\, xy = 0 \,\}$ normal crossing divisor, N = red submanifold.



Euler-like vector fields and sections

- F. Bischoff, H. Bursztyn, H. Lima, and E. Meinrenken, *Deformation spaces and normal forms around transversals*, Compos. Math. **156** (2020), no. 4, 697–732.
- H. Bursztyn, H. Lima, and E. Meinrenken, *Splitting theorems for Poisson and related structures*, J. Reine Angew. Math. **754** (2019), 281–312.

Definition

Let N be a submanifold of M. A vector field v on M is *Euler-like* with respect to N if v is complete and, in a suitable tubular neighbourhood of N, v looks like the Euler vector field on the normal bundle $\mathcal{N}(M, N)$.

Definition

Let $\pi\colon A \to M$ be a Lie algebroid and let $N \to M$ be a submanifold. An *Euler-like* section of A along N is a section $\varepsilon \in \Gamma(A)$ such that $\varepsilon|_N = 0$ and $\mathbf{an}(\varepsilon)$ is an Euler-like vector field along N.

Euler-like sections: the case of normal crossing divisors I

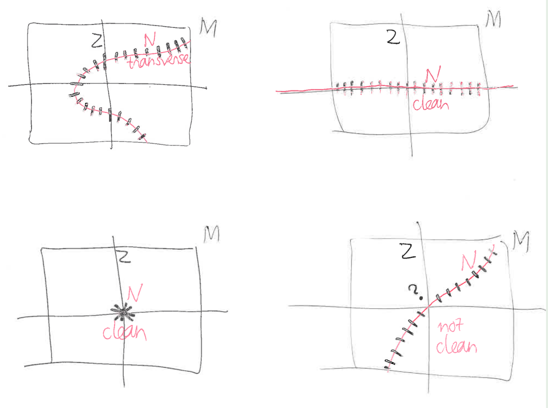
Let $Z \subset M$ be a normal crossing divisor, $A = TM(-\log Z)$. Let $N \subset M$ submanifold. An Euler-like section of A along N is an Euler-like vector field along N which is tangent to (every component of) Z.

Fact

An Euler-like section of $A = TM(-\log Z)$ along N exists iff N intersects A cleanly.

Example (revisited)

Let $M = \mathbf{R}^2$, $A = TM(-\log Z)$, where $Z = \{\, xy = 0 \,\}$ normal crossing divisor, N = red submanifold.



Utility of Euler-like sections, transverse case

"Quantitative" version of Dufour-Fernandes-Weinstein splitting theorem for Lie algebroids:

Theorem (Burstyn-Lima-Meinrenken)

*Let $A \to M$ be a Lie algebroid and $j \colon N \to M$ a submanifold **transverse** to A. Then an Euler-like section of A along N exists. Let U be the tubular neighbourhood of N associated with the Euler-like vector field $\mathbf{an}(\varepsilon)$. The flow of the Euler-like section gives a Lie algebroid isomorphism*

$$\pi^!_{M,N} j^! A \cong A|_U,$$

where $\pi_{M,N} \colon \mathcal{N}(M,N) \to N$ is the projection.

Utility of Euler-like sections, clean case

If a submanifold $j\colon N \longrightarrow M$ **cleanly** intersects a Lie algebroid $A \longrightarrow M$, an Euler-like section along N may not exist, and the splitting theorem may well fail. But:

Theorem (retraction theorem, LLSS)

Let $A \longrightarrow M$ be a Lie algebroid and let $j\colon N \longrightarrow M$ be a submanifold that cleanly intersects A. Suppose there exists an Euler-like section ε of A along N. Let U be the tubular neighbourhood of N associated with the Euler-like vector field $\mathbf{an}(\varepsilon)$. *The flow of ε gives rise to a deformation retraction of $A|_U$ onto $j^! A$. In particular j induces a homotopy equivalence $\Omega^\bullet_A(U) \simeq \Omega^\bullet_{j^! A}(N)$.*

This suffices for the purpose of the Darboux-Moser-Weinstein theorem.

Lie algebroid homotopies

Homotopies of Lie algebroid paths were introduced by

- M. Crainic and R. Fernandes, *Integrability of Lie brackets*, Ann. of Math. (2) **157** (2003), no. 2, 575–620.

More generally:

Definition

A *homotopy of Lie algebroids* is a Lie algebroid morphism $\varphi\colon T[0,1] \times A \longrightarrow B$, where $A \longrightarrow M$ and $B \longrightarrow N$ are Lie algebroids. We say φ is a homotopy *between* φ_0 *and* φ_1.

Just as in ordinary de Rham theory:

Fact

If two Lie algebroid morphisms $\varphi_0, \varphi_1\colon A \longrightarrow B$ are homotopic, then the induced maps $\varphi_0^*, \varphi_1^*\colon \Omega_B^\bullet(N) \longrightarrow \Omega_A^\bullet(M)$ are homotopic.

Lie algebroid retractions

Definition

Let $j\colon N \to M$ be a submanifold of M which cleanly intersects a Lie algebroid $A \to M$. A *deformation retraction of A onto $j^!A$* is a homotopy $\varrho\colon T[0,1] \times A \to A$ such that $\varrho_0 = \mathrm{id}_A$, $\varrho_1(A) = j^!A$, and $\varrho_1|_{j^!A} = \mathrm{id}_{j^!A}$.

Fact

Suppose there exists a deformation retraction $\varrho\colon T[0,1] \times A \to A$ onto $j^!A$. Then j induces a homotopy equivalence

$$j_\sharp^*\colon \Omega_A^\bullet(M) \xrightarrow{\simeq} \Omega_{j^!A}^\bullet(N)$$

with inverse ϱ_1^*.

The End