Mathematics
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Generalized orbital varieties and MV cycles

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Lie theory and integrable systems in symplectic and Poisson geometry

Matrix facts

Proposition

If

$$A = \begin{bmatrix} C & v & w \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{Mat}(N)$$

has Jordan type λ *and* C *has Jordan type* λ' *such that* $\lambda - \lambda'$ *is two boxes in columns* $l \leq r$*, then*

$$B = \begin{bmatrix} C & v \\ 0 & 0 \end{bmatrix} \in \mathrm{Mat}(N-1)$$

has Jordan type λ'' *such that* $\lambda - \lambda''$ *is the box in column* r *and* $r \neq l$*.*

Roughly, there is a natural order on Jordan types of submatrices of matrices like A.

In nature

These types of matrices show up when we consider the action of the indeterminate t on the affine Grassmannian

$$\mathcal{G}r = G(\mathcal{K})/G(\mathcal{O}) \qquad \mathcal{O} = \mathbb{C}[[t]], \mathcal{K} = \mathbb{C}((t))$$

in type A.

In nature

If L is a free submodule of the vector space $\mathcal{K}^m$ such that $\mathcal{K} \otimes_{\mathcal{O}} L \cong \mathcal{K}^m$ then we call it a rank m lattice.

We can identify the affine Grassmannian of GL_m with the set of all rank m lattices.

Moreover, to a weight $\mu \in \mathbb{Z}^m$ of GL_m we can associate an orbit $\mathcal{G}r_\mu \subset \mathcal{G}r$, and if $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \neq 0) \vdash N$ then any $L \in \mathcal{G}r_\mu$ is a sublattice of the standard lattice $L_0 = \mathcal{O}^m$ and the matrix[1]

$$\left[t|_{L_0/L} \right]'$$

is a $\mu \times \mu$ block matrix.

The collection of all such matrices is denoted $\mathbb{T}_\mu$ and the subset $\mathbb{T}_\mu \cap \mathfrak{n}$ is a particular subset of $(C, v, w, 1)$ matrices, assuming $\mu_m \geq 2$.

[1] in the basis $([e_1], [te_1], \ldots, [t^{\mu_1 - 1} e_m], \ldots, [e_m], [te_m], \ldots, [t^{\mu_m - 1} e_m])$

Examples

If $\mu = (3,2,2)$ and $L \in \mathcal{G}r_\mu$ then

$$\left[t\Big|_{L_0/L}\right]' = \left[\begin{array}{ccc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ * & * & 0 & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ * & * & 0 & * & * & * & * \end{array}\right]$$

with $*$s denoting unconstrained entries, and $\mathbb{T}_\mu \cap \mathfrak{n}$ comprises

$$\left[\begin{array}{ccc|cc|c|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} C & v & w \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right]$$

If $\mu = (1,1,\ldots,1)$ then $\mathbb{T}_\mu = \mathsf{Mat}(m)$.

The Mirković–Vybornov isomorphism

If $\lambda \geq \mu$ then $\mathbb{T}_\mu$ is a transverse slice to the conjugacy class of Jordan type λ, denoted $\mathbb{O}_\lambda$, and

$$\overline{\mathbb{O}}_\lambda \cap \mathbb{T}_\mu \cong \overline{\mathcal{G}r^\lambda} \cap \mathcal{G}r_\mu$$

where $\mathcal{G}r^\lambda$ is the $G(\mathcal{O})$ orbit associated to λ.

The restricted Mirković–Vybornov isomorphism

Corollary

$$\overline{\mathbb{O}}_\lambda \cap \mathbb{T}_\mu \cap \mathfrak{n} \cong \overline{\mathcal{G}r^\lambda} \cap S^\mu_-$$

where S^μ_- is the $U_-(\mathcal{K})$ orbit associated to μ.

Here $U_- \subset GL_m$ is the subgroup of unipotent lower triangular matrices.

"The left-hand side is made of $N \times N$ matrices while the right-hand side is made of $m \times m$ matrices."

Goal

Give a decomposition of the left-hand side $\overline{\mathbb{O}}_\lambda \cap \mathbb{T}_\mu \cap \mathfrak{n}$ by "generalized orbital varieties" indexed by semi-standard Young tableau matching the decomposition of (almost the right-hand side) $\overline{\mathcal{G}r^\lambda \cap S^\mu_-}$ into its irreducible components, the MV cycles.

Spaltenstein recipe

To tableau $\tau \in SSYT(\lambda)_\mu$ we can associate a matrix $A \in \mathbb{T}_\mu \cap \mathfrak{n}$ such that for all $1 \leq i \leq m$ the upper submatrix made of the first $i \times i$ blocks has Jordan type $\lambda^{(i)} = $ shape of $\tau\big|_{1,2,\ldots,i}$

$$A_{(i)} \in \mathbb{O}_{\lambda^{(i)}}$$

By example,

$$\tau = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \supset \tau\big|_{1,2} = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \supset \tau\big|_{1} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$$

defines a matrix

$$\left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a & b & c \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & d \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad a, d = 0 \text{ and } b, c \neq 0$$

Slower Spaltenstein recipe

We can slow the process down

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \supset \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \supset \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \supset \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \supset \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

so that a tableau of size N yields $N \geq m$ shapes

$$\lambda^{(3,1)} \supset \lambda^{(2,2)} \supset \lambda^{(2,1)} \supset \lambda^{(1,2)} \supset \lambda^{(1,1)}$$

and define a matrix A'

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

such that

$$A'_{(i,k)} \in \mathbb{O}_{\lambda^{(i,k)}}$$

Same result

We can use the $(C, v, w, 1)$ matrix fact to show that N conditions define the same matrices as m conditions, and slowing the recipe down does not alter the result

$$\{A \in \mathbb{T}_\mu \cap \mathfrak{n} \big| A_{(i)} \in \mathbb{O}_{\lambda^{(i)}}\} = \{A \in \mathbb{T}_\mu \cap \mathfrak{n} \big| A_{(i,k)} \in \mathbb{O}_{\lambda^{(i,k)}}\}.$$

We denote this set $\mathring{X}_\tau$.

Generalized orbital varieties

By induction on the restriction map

$$\mathring{X}_\tau \to \mathring{X}_{\tau - \boxed{m}}$$

we can show that $\mathring{X}_\tau$ has unique irreducible component of top dimensional.

$X_\tau := \overline{\mathring{X}_\tau^{top}}$ is called a generalized orbital variety, and

$$\overline{\mathbb{O}}_\lambda \cap \mathbb{T}_\mu \cap \mathfrak{n} = \bigcup_\tau X_\tau$$

Theorem

The closure of the image of X_τ under the Mirković–Vybornov isomorphism is an MV cycle.

Moreover, its Lusztig datum is equal to that of τ.

Here Lusztig data are certain combinatorial fingerprints of crystal bases.

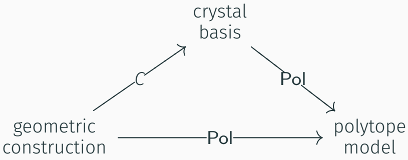
Application

We use the information of the ideal of an orbital variety corresponding to a given MV cycle to compare to perfect bases.

- the MV basis, $\{b_Z\}$, indexed by MV cycles Z
- the dual semicanonical basis, $\{\xi_M\}$, indexed by modules M

Application

Both bases are $B(\infty)$ crystal bases, with compatible polytope models, i.e.



such that $\mathrm{Pol}(Z) = \mathrm{Pol}(M)$ whenever $C(b_Z) = C(\xi_M)$.

Kamntizer–Knutson:

$$\mathrm{Pol}(Z) = \mathrm{Pol}(M) \overset{?}{\Longrightarrow} b_Z = \xi_M$$

Relaxed:

$$\mathrm{Pol}(Z) = \mathrm{Pol}(M) \overset{?}{\Longrightarrow} D(b_Z) = D(\xi_M)$$

Counterexample

Let (Y, Z) be such that

$$\tau(Y) = \tau(Z) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 5 & 5 \\ \hline 2 & 2 & 6 & 6 \\ \hline 3 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

then X_τ is 16 dimensional generated in degrees 1,2,3, and 6, and

$$\sum_{\underline{\mathrm{i}}} \chi(F_{\underline{\mathrm{i}}} M) D_{\underline{\mathrm{i}}} \neq \varepsilon_0(Z)$$

implying $D(\xi_M) \neq D(b_Z)$ so $\xi_M \neq b_Z$.

Thank you