Normal forms for Higgs fields over a formal 1d disc

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(Co-)Higgs fields

Let X be a smooth curve over $\mathbb{C}$.
Let V be a holomorphic vector bundle over X.
A tensor $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X^*)$ is called a **Higgs** field on V.
A tensor $\phi \in H^0(X, End(V) \otimes \mathcal{T}_X)$ is called a **co-Higgs** field on V.
If $X \subset \mathbb{C}$, then $\mathcal{T}_X^* \cong \mathcal{T}_X \cong \mathcal{O}_X$, and so, co-Higgs = Higgs.

Spectral correspondence

Let $\phi \in H^0(X, End(V) \otimes L)$, where L is a line bundle over the curve X.

$$\chi_\phi(\lambda) = \det(\lambda \mathrm{Id} - \phi) =$$
$$= \lambda^r - s_1(\phi)\lambda^{r-1} + \ldots + (-1)^{r-1} s_{r-1}(\phi)\lambda + (-1)^r s_r(\phi),$$

where $r = \mathrm{rk}\ V$, $s_i(\phi) \in S^i(L)$, λ is the tautological section of the pullback p^*L, where $p : L \to X$.
$\Sigma = \{\chi_\phi(\lambda) = 0\} \subset L$ is called the **spectral curve** of ϕ.

Theorem (Beauville-Narasimhan-Ramanan)

$$\left\{ \begin{array}{l} V: \text{ vector bundle on } X, \\ \phi \in H^0(X, End(V) \otimes L) \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \Sigma \subset L, \\ \text{rank 1, torsion-free} \\ \text{sheaf } \mathcal{F} \text{ on } \Sigma \end{array} \right\}$$

$$V = p_*(\mathcal{F})$$
$$\phi = p_*(\text{multiplication by } \lambda)$$

Normal forms for (co-)Higgs fields over a formal 1d disc

Let $\mathcal{U} = \operatorname{Spec} \mathbb{C}[[x]]$ be a formal 1d disc.
Let $V = \mathcal{O}_{\mathcal{U}}^{\oplus r}$, and $\phi \in \operatorname{End}(V)$ a (co-)Higgs field on V.
Assuming spectral curve Σ of ϕ is fixed, is there a normal form for ϕ?
E.g. if $\Sigma = \{y^r = x\}$, then ϕ is isomorphic to

$$\left(\begin{array}{ccccc|c} & & & & & x \\ \hline 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 1 & \end{array}\right)$$

Is there an analogous statement for singular Σ?

Why consider singular spectral curves?

Let V be a holomorphic vector bundle over a curve X.
A $\mathbb{C}^*$-invariant Poisson structure σ on $V \implies$ a co-Higgs filed ϕ on V.

$$\phi : \mathcal{T}^*_X \longrightarrow End_0(V)$$

$$\alpha \longmapsto \begin{array}{c}\sigma\text{-Hamiltonian vector field} \\ \text{of the pullback of } \alpha \text{ to } \mathbb{P}(V)\end{array}$$

A Poisson structure σ on $\mathbb{P}(V) \implies$ a zero-trace co-Higgs filed ϕ on V.

Theorem (M.)

Let V be a rank > 2 holomorphic vector bundle over a connected, simply connected curve $X \subseteq \mathbb{P}^1$.
Let σ be either a Poisson structure on $\mathbb{P}(V)$, or a $\mathbb{C}^*$-invariant Poisson structure on V.
Then the spectral curve of the co-Higgs field on V induced by σ, if it is connected, must be **singular**.

Frame Title

Let $\mathcal{U} = \text{Spec } \mathbb{C}[[x]]$ be a formal 1d disc.
Let $V = \mathcal{O}_{\mathcal{U}}^{\oplus r}$, and $\phi \in \text{End}(V)$ a (co-)Higgs field on V.
Fix a (singular) spectral curve Σ of ϕ. What does ϕ look like?
Spectral correspondence:

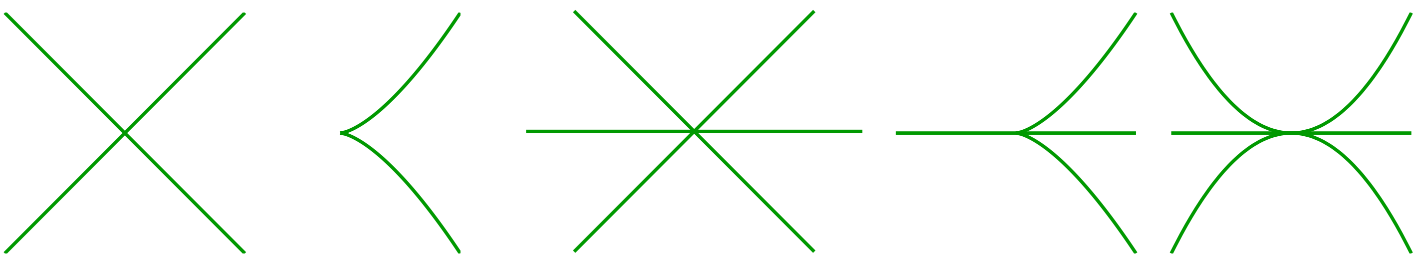
$$\text{isomorphism classes of } \phi \xleftrightarrow{1-1} \text{torsion-free rank 1 sheaves on } \Sigma$$

Classification of torsion-free rank 1 sheaves is done for the curves with ADE singularities (Drozd-Roiter, Jacobinski, 1967), T_{pq} singularities (e.g. Schappert 1987), and a few more cases in Arnold's classification of singularities (e.g. "unimodal", "bimodal").

ADE and T_{pq} singularities

$$
\begin{aligned}
&(A_n) && x^2 - y^{n+1} && (n \geq 1),\\
&(D_n) && x^2 y - y^{n-1} && (n \geq 4),\\
&(E_6) && x^3 - y^4, &&\\
&(E_7) && x^3 y - y^3, &&\\
&(E_8) && x^3 - y^5. &&
\end{aligned}
$$

$$
\begin{aligned}
&(T_{36}) && y(y - x^2)(y - ax^2), && a \in \mathbb{C} \setminus \{0, 1\},\\
&(T_{44}) && yx(y - x)(y - ax), && a \in \mathbb{C} \setminus \{0, 1\},\\
&(T_{pq}) && (x^{p-2} - y^2)(x^2 - ay^{q-2}), && a \in \mathbb{C},\ 1/p + 1/q < 1/2.
\end{aligned}
$$



$A_1 : y^2 = x^2$ $\quad A_2 : y^2 = x^3$ $\quad D_4 : y^3 = x^2 y$ $\quad E_7 : y^3 = x^3 y$ $\quad T_{36} : y^3 = x^4 y$

Higgs fields with an A_n singularity

Let $\Sigma = \{y^{n+1} = x^2\}$.

Let $\phi \in \mathrm{End}(\mathcal{O}_{\mathcal{U}}^{\oplus(n+1)})$ whose spectral curve is Σ.

Then ϕ is isomorphic one of the following:

$$\left(\begin{array}{cccccccc|c} & & & & & & & & x^2 \\ \hline 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \end{array}\right)$$

corresponds to the sheaf $\mathcal{O}_\Sigma$

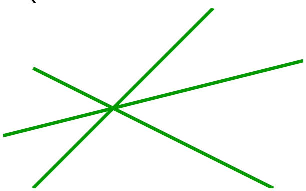
$$\left(\begin{array}{ccccccc|c} & & & & & & & x \\ \hline 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & x & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & 1 & \end{array}\right)$$

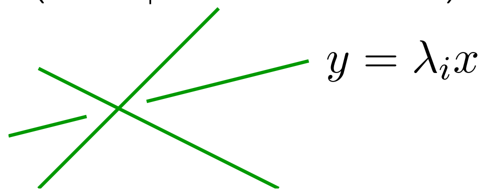
corresponds to $p_*\mathcal{O}_{\widetilde{\Sigma}}$,
where $p : \widetilde{\Sigma} \to \Sigma$ is a partial
resolution of singularities;
$\widetilde{\Sigma}$ is either smooth
or has an A_{n-2k} for some k.

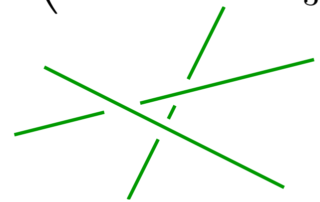
Higgs fields with a D_4 singularity

Let $\Sigma = \{(y - \lambda_1 x)(y - \lambda_2 x)(y - \lambda_3 x) = 0\}$, $\lambda_i \in \mathbb{C}$, $\lambda_i \neq \lambda_j$.
Let $\phi \in \mathrm{End}(\mathcal{O}_{\mathcal{U}}^{\oplus 3})$ whose spectral curve is Σ.
Then ϕ is isomorphic to exactly one of:

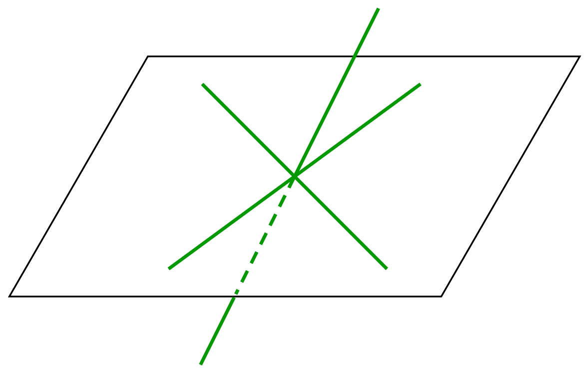
$$\begin{pmatrix} \lambda_1 x & & \\ 1 & \lambda_2 x & \\ & 1 & \lambda_3 x \end{pmatrix} \qquad \left(\begin{array}{c|cc} \lambda_i x & & \\ \hline & \lambda_{i+1} x & \\ & 1 & \lambda_{i+2} x \end{array}\right) \qquad \begin{pmatrix} \lambda_1 x & & \\ & \lambda_2 x & \\ & & \lambda_3 x \end{pmatrix}$$







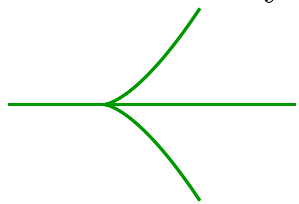
$$\left.\begin{matrix} \begin{pmatrix} \lambda_1 x & & \\ x & \lambda_2 x & \\ & 1 & \lambda_3 x \end{pmatrix} \\[2em] \begin{pmatrix} \lambda_1 x & & \\ 1 & \lambda_2 x & \\ & x & \lambda_3 x \end{pmatrix} \end{matrix}\right\}$$



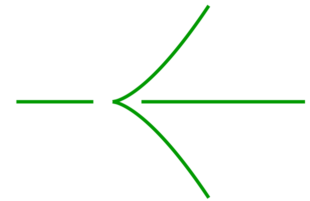
Higgs fields with an E_7 singularity

Let $\Sigma = \{y^3 = x^3 y\}$. Let $\phi \in \mathrm{End}(\mathcal{O}_{\mathcal{U}}^{\oplus 3})$ whose spectral curve is Σ. Then ϕ is isomorphic to exactly one of:

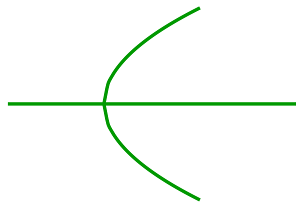
$$\left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x^3 \\ & 1 & \end{array}\right)$$



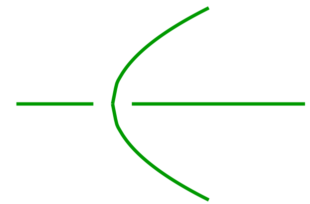
$$\left(\begin{array}{c|cc} 0 & & \\ \hline & & x^3 \\ & 1 & \end{array}\right)$$



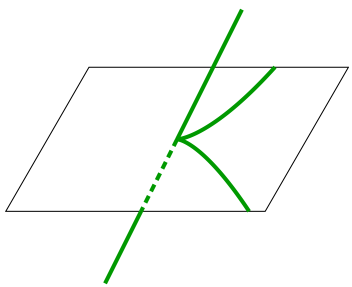
$$\left(\begin{array}{c|cc} 0 & & \\ \hline x & & x^2 \\ & x & \end{array}\right)$$



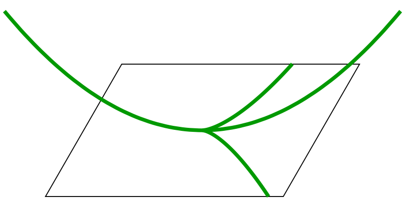
$$\left(\begin{array}{c|cc} 0 & & \\ \hline & & x^2 \\ & x & \end{array}\right)$$



$$\left.\begin{array}{c} \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x \\ & x^2 & \end{array}\right) \\ \left(\begin{array}{c|cc} 0 & & \\ \hline x^2 & & x^3 \\ & 1 & \end{array}\right) \end{array}\right\}$$



$$\left.\begin{array}{c} \left(\begin{array}{c|cc} 0 & & \\ \hline 1 & & x^2 \\ & x & \end{array}\right) \\ \left(\begin{array}{c|cc} 0 & & \\ \hline x & & x^3 \\ & 1 & \end{array}\right) \end{array}\right\}$$



Conclusion

Using the classification of torsion-free rank 1 sheaves over ADE singularities singularities, available in the literutare on Cohen-Macaulay modules over Cohen-Macaulay curves, we deduce the normal forms for (co-)Higgs fields over formal 1d disc whose spectral curve has such a singularity (Appendix of the PhD thesis: `http://blog.math.toronto.edu/GraduateBlog/2020/05/15/departmental-phd-thesis-exam-mykola-matviichuk/`)

Thank you

THANK YOU!