SU(2) Representations of the Fundamental Group of a Genus 2 Oriented 2-manifold

Lisa Jeffrey,

Dept. of Mathematics

University of Toronto

http://www.math.toronto.edu/~jeffrey

Joint work with Nan-Kuo Ho, Paul Selick and Eugene Xia

arXiv:2005.07390

June 7, 2020

I. Introduction

- Let Σ be a compact two-dimensional orientable manifold of genus 2 (in other words a double torus).

- After puncturing the surface, the fundamental group is the free group on four generators.

- We consider the representations of this fundamental group into $SU(2)$ for which the loop around the puncture is sent to $-I$.

  This space is well studied by Desale-Ramanan (1976), and has been identified with the space of planes in the intersection of two quadrics in a Grassmannian.

- Define $M = \mu^{-1}(-I)$ where μ is the product of commutators.

- Define $A = M/G$ where $G = SU(2)$ acts by conjugation.

- Special case of Atiyah and Bott 1983, who found that these spaces of conjugacy classes of representation of the fundamental group were torsion free and computed their Betti numbers.
- The ring structure of the cohomology was discovered by Thaddeus (1992) using methods from mathematical physics and algebraic geometry.
- In this special case, we recover Atiyah and Bott's result and also Thaddeus' result.
- Our main tool is the Mayer-Vietoris sequence.

Results:

- Cell decomposition and ring structure of the space of commuting elements of SU(2)

  (Previously the cohomology groups were identified by Adem and Cohen 2006; a cell decomposition of the suspension of this space was studied by Baird, Jeffrey, Selick 2009).

- Cohomology groups of $M = \mu^{-1}(-I)$; cohomology ring of $M' = M/M_U$ where M_U is the subset of elements where at least one element is in the center of $SU(2)$

- New calculation of the cohomology of the space A of conjugacy classes of representations.

  Cohomology groups: Atiyah-Bott 1983

  Cohomology ring: Thaddeus 1992 (using Verlinde formula)

- Identification of the transition functions of the principal $SU(2)$ bundle $M \to A$

II. The space of commuting elements

- The space of commuting elements is $\mathcal{T} := Comm^{-1}(I)$ where $Comm : G \times G \to G$ is the commutator.

  The structure of the cohomology of $\mathcal{T}$ as groups was discovered by Adem-Cohen (2006).

  The cell decomposition of the suspension of $\mathcal{T}$ was worked out by Baird-Jeffrey-Selick (2009) and Crabb (2011).

- We write the elements of $SU(2)$ as quaternions

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \longleftrightarrow z + wj$$

  Let T be the maximal torus of G (the space of diagonal unitary matrices of rank 2 with determinant 1). This is isomorphic to the circle group $U(1)$.

For all $g \in G$ $\exists$ $\theta \in [0, \pi]$ s.t. (as quaternions) $g = he^{i\theta}h^{-1}$ for some $h \in G$. This occurs if and only if $\text{Trace}(g) = e^{i\theta} + e^{-i\theta} = 2\cos(\theta)$. The group G is foliated by its conjugacy classes, which are parametrized by the value of the trace map.

- So $G \times G$ is foliated by the values of the trace of the commutator map:

$$G \times G = \cup_{\theta \in [0,\pi]} W_\theta$$

where

$$W_\theta = \{(g, h) \mid [g, h] \sim e^{i\theta}\}.$$

- Define

$$X_\theta = \{(g, h) \mid [g, h] = e^{i\theta}\}$$

- Define also

$$W_{[a,b]} = \cup_{\theta \in [a,b]} W_\theta$$

(similarly $X_{[a,b]}$).

Theorem [Meinrenken]: For $\theta \neq 0$,

$$X_\theta = PSU(2) = SO(3) = \mathbf{R}P^3 := \mathcal{H}$$

There is a homeomorphism from X_θ to $\mathcal{H}$, where T acts on X_θ by conjugation and acts on $\mathcal{H}$ by left translation. $\square$

By writing down an explicit T-homeomorphism, we show that there is a T-equivariant homeomorphism from X_θ to $\mathcal{H}$.

III. RETRACTIONS

- The space $\mathcal{T} = X_0 = \{g, h | [g, h] = 1\}$ is the space of commuting pairs in $SU(2)$.

  Theorem: There is a deformation retraction from $X_{[0,\pi)}$ to X_0.

  Recall the following theorem of Milnor:

  Theorem (Milnor, *Morse Theory*) If $f : M \to \mathbf{R}$ is smooth and c is an isolated critical value of f, and f has no critical values in $(c, d]$, then $f^{-1}(c)$ is a deformation retract of $f^{-1}([c, d])$.

  We apply this theorem to the trace function on $X_{[0,\pi)}$. The extreme value is $\mathrm{Trace}(g) = 2$, in other words $\theta = 0$.

  Instead of using Milnor's theorem, we use the gradient flow for the trace function.

  Theorem: The flow lines for the vector field $\nabla(\mathrm{T}race)$ are closed, and every point of X_θ is the endpoint of a flow line emanating from X_u for some $u > 0$.

- However, we cannot get a closed form solution for the equation of the flow lines.
- We have found a different retraction which is explicit, and this allows us to show that it is T-equivariant.

IV. Cohomology of Commuting Pairs

- Baird, Jeffrey, Selick (2009) gave the cohomology of the suspension of $\mathcal{T}$, showing that this suspension is equivalent to the suspension of $S^3 \vee S^3 \vee S^2 \vee \Sigma^2 \mathbf{R}P^2$.

- Instead,

$$SU(2) \times SU(2) = W_{[0,\pi]} = W_{[0,\pi)} \cup W_{(0,\pi]}$$

$$W_{[0,\pi)} \simeq W_0 = X_0 = \mathcal{T}$$

$$W_{(0,\pi]} \simeq W_\pi = X_\pi = \mathbf{R}P^3$$

$$W_{(0,\pi)} = (0,\pi) \times W_{\pi/2} = (0,\pi) \times (\mathbf{R}P^3 \times S^2)$$

- By Mayer-Vietoris, we are able to compute the cohomology of $\mathcal{T}$ as a ring. It turns out that all cup products are 0.

- We show that $X_0 \simeq S^3 \vee S^3 \vee S^2 \vee \Sigma^2 \mathbf{R}P^2$.

See also the 2016 PhD thesis of Trefor Bazett.

V. Atiyah Space

- Let

$$M = \mu^{-1}(-I).$$

This level set is a 9-manifold.

- The space $A := M/G$ (where G acts on M by conjugation). The center of G acts trivially, so we have a free $SO(3)$-action.

The space $A = M/G$ is a free $SO(3)$ bundle.

Theorem (*Atiyah-Bott 1983*)

$$H^*(A) = \mathbf{Z}, q = 0, 2, 4, 6$$

$$= \mathbf{Z}^4, q = 3$$

All the other groups are 0.

- We have

$$A_\theta = \{(x, y, x', y') \in G \mid [x, y][x', y'] = -I, [x, y] \simeq e^{i\theta}\}/SO(3)$$

$$A_0 = (X_0 \times X_\pi)/SO(3) = X_0 = \mathcal{T}$$

$$A_\pi = (X_\pi \times X_0)/SO(3) = X_0 = \mathcal{T}$$

- For $\theta \in (0, \pi)$,

$$A_\theta = (X_\theta \times X_{\pi-\theta})/SO(3) = \mathbf{R}P^3 \times (\mathbf{R}P^3/T) = \mathbf{R}P^3 \times S^2$$

- We can write an explicit retraction for this:

$$A_{[0,\pi)} \simeq A_0 \simeq \mathcal{T}.$$

where $A_{[0,\pi)} := \cup_{\theta\in[0,\pi)} A_\theta$.

$$A = A_{[0,\pi)} \cup_{A_{(0,\pi)}} A_{(0,\pi]} \simeq \mathcal{T} \times_{\mathbf{R}P^3\times S^2} \mathcal{T}.$$

- Mayer-Vietoris gives the cohomology groups of A as above.

VI. The 9-Manifold

Recall we defined $M = \mu^{-1}(-I)$.

Then

$$M = \cup_{\theta \in [0,\pi]} M_\theta,$$

where

$$M_\theta = \{(x, y, x', y') \in M \mid [x, y] \sim e^{i\theta}\}.$$

Lemma. The bundles

$$M_{[0,\pi)} \to A_{[0,\pi)}$$

and

$$M_{(0,\pi]} \to A_{(0,\pi]}$$

are trivial.

(This implies there is a local trivialization of $M \to A$ over $A = A_{[0,\pi)} \cup A_{(0,\pi]}$.)

Theorem: The transition function is given by

$$A_{(0,\pi)} \simeq A_{\pi/2} = (X_{\pi/2} \times X_{\pi/2})/T = (\mathbf{R}P^3 \times \mathbf{R}P^3)/T \to \mathbf{R}P^3.$$

where the last map is given by $(g, h) \mapsto g^{-1}h$. This is well defined because T acts by left multiplication, where we have made use of the fact that our homeomorphism $X_{\pi/2} \to \mathbf{R}P^3$ is a T-map with respect to the conjugation action on $X_{\pi/2}$ and left multiplication on $\mathbf{R}P^3$.

VII. Prequantum Line Bundle

Let $A' = A/A_U$ where A_U is the subset of x, y, x', y' for which at least one of x, y, x', y' is $\pm I$. We note that $A'_\theta = A_\theta$ for $\theta \neq 0, \pi$.

- Let $\mathcal{L}$ be the total space of the prequantum $U(1)$ bundle over A'. Let $\mathrm{proj}_L : \mathcal{L} \to A'$ be the projection map. The space $\mathcal{L}$ may be formed as a union of the two open sets $\mathrm{proj}_L^{-1}(A'_{[0,\pi)})$ and $\mathrm{proj}_L^{-1}(A'_{(0,\pi]})$. These sets intersect in a subset $\mathrm{proj}_L^{-1}(A'_{(0,\pi)})$. This subset is isomorphic to $\mathbf{R}P^3 \times \mathbf{R}P^3$.
- So we are able to identify its cohomology groups.
- We examine the Mayer-Vietoris sequence associated to the above decomposition of $\mathcal{L}$. We first do this with $\mathbf{Z}/2\mathbf{Z}$ coefficients and obtain

$$H^q(\mathcal{L}; (\mathbf{Z}/2\mathbf{Z})) = \mathbf{Z}/(2\mathbf{Z}), q = 0, 3, 4, 7$$

and 0 for all other values of q.

- Then we study the Mayer-Vietoris sequence with integer coefficients. The sequence for

$$0 \to \mathrm{coker}(\delta) \to H^4(\mathcal{L}) \to \ker(\delta) \to 0$$

is

$$0 \to \mathbf{Z}/(2\mathbf{Z}) \to H^4(\mathcal{L}) \to \mathbf{Z}/(2\mathbf{Z}) \to 0$$

Hence $H^4(\mathcal{L})$ has four elements. Because we have already computed $H^4(\mathcal{L}; \mathbf{Z}/(2\mathbf{Z}))$ and this has one element, it follows that $H^4(\mathcal{L}; \mathbf{Z}) = \mathbf{Z}/(4\mathbf{Z})$.

- The cohomology of the total space of the prequantum line bundle is

$$H^q(\mathcal{L}; \mathbf{Z}) = \mathbf{Z}, \qquad q = 0, 7;$$

$$\mathbf{Z}/4\mathbf{Z}, \qquad q = 4.$$

For all other values of q,

$$H^q(\mathcal{L};\mathbf{Z}) = 0.$$

We may then make the following deduction:

- **Corollary:**

  The ring structure of $H^*(A)$ is

  $$H^*(A) =< 1, x, s_1, s_2, s_3, s_4, y, z >$$

  where the degrees of x,y and z are respectively 2, 4, 6 and the degree of the s_j are 3.

  The relations are

  $$x^2 = 4y, xy = s_1 s_3 = s_2 s_4 = z$$

  and all other intersection pairings are 0.

- The cohomology ring $H^*(A')$ is the same, except with no generators in degree 3.
- These relations were first shown by Thaddeus 1992.

VIII. COHOMOLOGY OF 9-MANIFOLD

We make the following definition:

$$M' = M/M_U$$

where M_U is the subset of M where at least one of x, y, x', y' is $\pm I$.

Since we know the transition function for the Mayer-Vietoris sequence, we can deduce

$$\begin{aligned} H^q(M') = \mathbf{Z}, &\qquad q = 0, 2, 7, 9 \\ \mathbf{Z}/(4\mathbf{Z}), &\qquad q = 4, 6 \end{aligned}$$

and all others are 0.

With some extra work, we can also get the ring structure of $H^*(M')$, and

$$H^q(M) = H^q(M') \oplus R$$

where $R^3 = R^6 = \mathbf{Z}^4, \quad R^5 = (\mathbf{Z}/2\mathbf{Z})^4$ and all the rest are 0.

IX. WALL'S THEOREM

- We conclude by matching up our results with the work of Wall (1966) toward classifying 6-manifolds:

  Theorem (C.T.C. Wall, 1966):

  Let Y be a 6-manifold with $H^q(Y) = \mathbf{Z}$ for $q = 0, 2, 4, 6$ and $H^3(Y) = \mathbf{Z}^{2r}$. Then Y is the connected sum of some 6-manifold Y' with the r-fold connected sum $(S^3 \times S^3)^{\# r}$, where $H^q(Y') = \mathbf{Z}, \quad q = 0, 2, 4, 6$ and 0 for all other q.

- In our case

$$A = A' \#(S^3 \times S^3) \#(S^3 \times S^3)$$

  where A' was previously defined as $A' = A/A_U$ where A_U is the subset of x, y, x', y' for which at least one of x, y, x', y' is $\pm I$.

- Note that part of Wall's results is that there exists a smooth manifold homeomorphic to A'.