# Symplectic cohomological rigidity through toric degenerations

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Lie theory and integrable systems in symplectic and Poisson geometry

# Symplectic toric manifolds

A **symplectic toric manifold** is

- a $2n$-dimensional closed, connected manifold $M$;
- an integral symplectic form $\omega$;
- a faithful $(S^1)^n$ action;
- a moment map $\mu\colon M \to \mathbb{R}^n$, i.e., $\iota_{\xi_j}\omega = -d\mu_j$ for all $1 \le j \le n$.

The **moment polytope** $\Delta := \mu(M)$ is a convex polytope.

**Facts:** [Delzant]

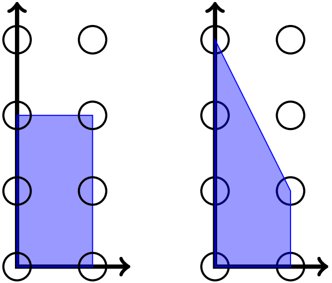
- $M, M'$ are equivariantly symplectomorphic exactly if $\Delta = \Delta' + c$.
- $M$ is a projective variety, i.e., $M \hookrightarrow \mathbb{P}^N$.

# Hirzebruch surfaces

Consider the **Hirzebruch surface** $\Sigma_m := \mathbb{P}(\mathbb{C} \oplus \mathcal{O}(-m)) \to \mathbb{P}^1$:

- $\Sigma_m$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$.
- $\Sigma_m$ a symplectic toric manifold. (Given $\Sigma_m \hookrightarrow \mathbb{P}^N$.)
- The moment polytope of $\Sigma_m$ is a trapezoid.
- $H^*(M;\mathbb{Z}) = \mathbb{Z}[x_1, x_2]/\left(x_2^2, x_1^2 + m x_1 x_2\right)$.

**Examples:** $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\Sigma_2$.



# Bott manifolds

Define a **Bott manifold** $M := X_n$ inductively:

- Let $X_1 = \mathbb{P}^1$.
- Given a holomorphic line bundle $L \to X_{i-1}$,
  $X_i := \mathbb{P}(L \oplus \mathbb{C})$ is $\mathbb{P}^1$ bundle over $X_{i-1}$ $\forall$ $i$.

Properties:

- $M$ is a symplectic toric manifold. (Given $M \hookrightarrow \mathbb{P}^N$.)
- The moment poltyope $\Delta$ is combinatorially equivalent to $[0,1]^n$.
- $\Delta = \{p \in \mathbb{R}^n \mid \langle p, e_j \rangle \geq 0 \text{ and } \langle p, e_j + \sum_i A^i_j e_i \rangle \leq \lambda_j \ \forall j\}$
  for some strictly upper triangular integral matrix $A$ and $\lambda \in \mathbb{Z}^n$.
- $H^*(M;\mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n] / \big( x_i^2 + \sum_j A^i_j x_i x_j \big)$ and $[\omega] = \sum_i \lambda_i x_i$,
  where $x_j$ is dual to the preimage of $\Delta \cap \{\langle p, e_j + \sum_i A^i_j e_i \rangle = \lambda_j\}$.

# Cohomological rigidity

Let $\mathcal{F}$ be a family of manifolds.
Fix $M, M' \in \mathcal{F}$.

If $M, M'$ are diffeomorphic, then $H^*(M;\mathbb{Z}) \simeq H^*(M';\mathbb{Z})$ (as rings).

**Question:**
Does $H^*(M;\mathbb{Z}) \simeq H^*(M';\mathbb{Z})$ imply that $M, M'$ are diffeomorphic?

If the answer is YES, then $\mathcal{F}$ is **cohomologically rigid**.

**Example:**

- Surfaces are cohomologically rigid.
- Hirzebruch surfaces are cohomologically rigid.

# Cohomological rigidity for toric manifolds?

**Question:** (Masuda-Suh)
Are toric manifolds cohomologically rigid?

**Theorem (Masuda-Panov (2008), Choi-Masuda (2012))**

*Let $X, X'$ be Bott manifolds with $H^*(X;\mathbb{Q}) \simeq H^*(X';\mathbb{Q}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Q})$. If $H^*(X;\mathbb{Z}) \simeq H^*(X';\mathbb{Z})$, then $X, X'$ diffeomorphic.*

Cohomological rigidity holds in other special cases.
[Cho, Choi, Lee, Masuda, Panov, Park, Suh]

There are no known counterexamples.

# Symplectic cohomological rigidity

Let $\mathcal{G}$ be a family of symplectic manifolds.
Fix $(M, \omega), (M', \omega') \in \mathcal{G}$.

If $M, M'$ are symplectomorphic, there's an isomorphism $H^*(M; \mathbb{Z}) \to H^*(M'; \mathbb{Z})$ with $[\omega] \mapsto [\omega']$.

**Question:**
Does an isomorphism $H^*(M; \mathbb{Z}) \to H^*(M'; \mathbb{Z})$ with $[\omega] \mapsto [\omega']$ imply that $M$ is symplectomorphic to $M'$?

If the answer is YES, then $\mathcal{G}$ is **symplectically cohomologically rigid**.

**Example:**

- Symplectic surfaces are symplectically cohomologically rigid.
- So are symplectic Hirzebruch surfaces.

# Symplectic rigidity for toric manifolds?

**Question:**
Are symplectic toric manifolds symplectically cohomologically rigid?

**Theorem (McDuff, 2011)**

*If $M$ is a symplectic toric manifold and $H^*(M;\mathbb{Z}) \simeq H^*(\mathbb{P}^i \times \mathbb{P}^j;\mathbb{Z})$, then $M$ is symplectomorphic to $\mathbb{P}^i \times \mathbb{P}^j$.*

Other partial results. [Karshon, Kessler, Pinsonnault, McDuff]

# Our main results

## Theorem (Pabiniak-T)

*Let $M, M'$ be symplectic Bott manifolds with*
$H^*(M;\mathbb{Q}) \simeq H^*(M';\mathbb{Q}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Q})$.
*If there's an isomorphism $H^*(M;\mathbb{Z}) \to H^*(M';\mathbb{Z})$ with $[\omega] \mapsto [\omega']$,*
*then $M, M'$ are symplectomorphic.*

.

## Corollary

*If $M$ is a symplectic toric manifold and $H^*(M;\mathbb{Z}) \simeq H^*((\mathbb{P}^1)^n;\mathbb{Z})$, then $M$ is symplectomorphic to $(\mathbb{P}^1)^n$ with symplectic form $\omega_\lambda := \sum_i \lambda_i \pi_i^*(\omega_{FS})$.*

**Proof of corollary:**
By a result of Masuda and Panov, *M* is a symplectic Bott manifold.

**Note:** Strong rigidity also holds.

# Proof of the main theorem

The key step is to construct new symplectomorphisms:

Otherwise, the proof is similar to the smooth case.

**Proposition $\star$**

*Let $M$ and $M'$ be symplectic Bott manifolds.*
*Assume there exist $k < \ell$ and an isomorphism $H^*(M;\mathbb{Z}) \to H^*(M';\mathbb{Z})$ with $[\omega] \mapsto [\omega']$, $x_k \mapsto x'_k - \gamma x'_\ell$ for some $\gamma \in \mathbb{Z}$, and $x_i \mapsto x'_i$ for all $i \neq k$.*
*If $c := \frac{1}{2}\left(A^k_\ell + (A')^k_\ell\right) \geq 0$, then $M, M'$ are symplectomorphic.*

So we need to prove this proposition.

# Toric degenerations

Let $X \subset \mathbb{P}^N$ be a smooth projective variety.

Fix a local coordinate system on $X$.

There's an associated semigroup $S = \cup_{m>0}\{m\} \times S_m \subset \mathbb{Z} \times \mathbb{Z}^n$.

The **Okounkov body** is $\Delta := \overline{\operatorname{conv} \cup_{m>0} \frac{1}{m} S_m}$.

**Theorem (Harada-Kaveh, 2015)**

*Assume $S$ is finitely generated.*
$X_0 := \operatorname{Proj} \mathbb{C}[S]$ *is a projective toric variety with moment polytope* $\Delta$.
*There's a a continuous surjective map* $\Phi \colon X \to X_0$ *that's a symplectomorphism on an open dense subset of* $X$.

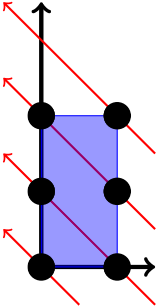
**Key observation:** If $X_0$ is smooth $\Phi$ is a symplectomorphism.

**Idea of proof:** Construct a toric degeneration of $X$,
i.e., a flat family $\pi \colon \mathcal{X} \to \mathbb{C}$ with generic fiber $X$ and $\pi^{-1}(0) = X_0$.
Lift a radial vector field to construct flow.

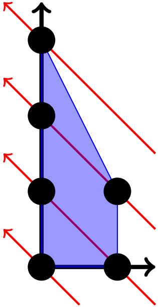
# The "slide" operator

Fix $w \in \mathbb{Z}^n \setminus \mathbb{Z}^n_{\geq 0}$. Construct the **slide** of $Q \subseteq \mathbb{Z}^n_{\geq 0}$ along $w$ by sliding each point as far as possible within $\mathbb{Z}^n_{\geq 0}$ in the direction $w$.

**Example:**



Slide in direction $-e_1 + e_2$.



**Claim:** Let $M, M'$ be symplectic toric manifolds with moment polytopes $\Delta, \Delta'$ that are equal to $\mathbb{R}^n_{\geq 0}$ near 0. If there exists $k < \ell$ and $c \geq 0$ with

$$\mathcal{S}_{-e_k + ce_\ell}(m\Delta \cap \mathbb{Z}^n) = m\Delta' \cap \mathbb{Z}^n \quad \forall m \in \mathbb{Z}_{>0},$$

then $M$ is symplectomorphic to $M'$.

**Recall:**

## Proposition $\star$

*Let $M$ and $M'$ be symplectic Bott manifolds.*
*Assume there exist $k < \ell$ and an isomorphism $H^*(M;\mathbb{Z}) \to H^*(M';\mathbb{Z})$ with $[\omega] \mapsto [\omega']$, $x_k \mapsto x'_k - \gamma x'_\ell$ for some $\gamma \in \mathbb{Z}$, and $x_i \mapsto x'_i$ for all $i \neq k$.*
*If $c := \frac{1}{2}\left(A^k_\ell + (A')^k_\ell\right) \geq 0$, then $M, M'$ are symplectomorphic.*

## Proof.

In the situation of Proposition $\star$,

$$\mathcal{S}_{-e_k+ce_\ell}(m\Delta \cap \mathbb{Z}^n) = m\Delta' \cap \mathbb{Z}^n \quad \forall m \in \mathbb{Z}_{>0} \quad \text{(or vice versa)}.$$

□