Symplectic geometry of toric degenerations for non-projective varieties

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Lie theory and integrable systems in symplectic and Poisson geometry

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This is based work with Jeremy Lane (McMaster/Fields):

Canonical bases and collective integrable systems

(on arxiv soon...)

If you are a (real) symplectic geometer, the world of smooth projective varieties is just too small.

I will illustrate this with a story.

Consider the Lie group $SU(n)$.

For a dominant weight λ of $SU(n)$, the coadjoint orbit $\mathcal{O}_\lambda$ of $SU(n)$ is has a symplectic form ω_λ.

Theorem (Guillemin-Sternberg)

There is a completely integrable torus action on $(\mathcal{O}_\lambda, \omega_\lambda)$.

1. There is a continuous map $\mu \colon M \to \mathbb{R}^N$, which is smooth on a dense subset of M.
2. On its smooth locus, μ is the moment map for a Hamiltonian $(S^1)^N$ action on (M, ω)
3. The action of $(S^1)^N$ is locally free on a dense subset, and $\dim M = 2N$.

Now, let K be any compact Lie group, and let λ be a dominant integral weight of K.

Theorem (Harada-Kaveh)

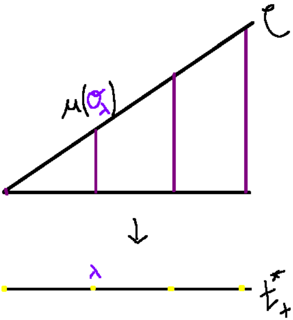
There is a completely integrable torus action on $(\mathcal{O}_\lambda, \omega_\lambda)$.

In fact:
There is a real convex polyhedral cone

$$\mathcal{C} \subset \mathbb{R}^N \times \mathfrak{t}^*_+,$$

so that

$$\mu(\mathcal{O}_\lambda) = \mathcal{C} \cap (\mathbb{R}^N \times \{\lambda\}).$$



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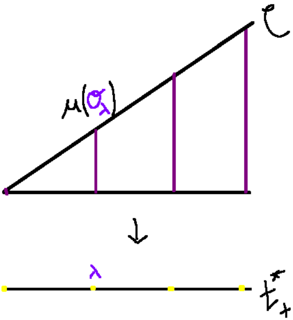

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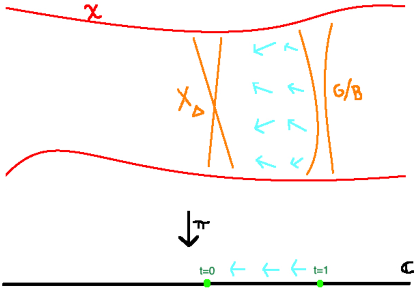
$(\mathcal{O}_\lambda, \omega_\lambda) \cong (G/B, \omega_\lambda) \hookrightarrow (\mathbb{P}^K, \omega_{FS})$

Find toric degeneration $\pi\colon \mathcal{X} \to \mathbb{C}$ of G/B to projective toric variety $X_\triangle$

$(\pi^{-1}(t) \cong G/B$ for $t \in \mathbb{C}^\times$,
and $\pi^{-1}(0) \cong X_\triangle)$



Integrate the vector field $-\frac{\nabla_{\Re\pi}}{||\nabla_{\Re\pi}||^2}$ to get a map $\pi^{-1}(1) \to \pi^{-1}(0)$.

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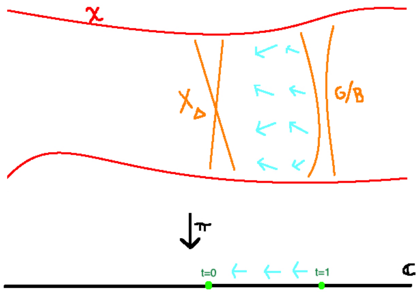

Because we insist
on everything being projective

If you are a (real) symplectic geometer, the world of smooth projective varieties is just too small.

However, in this case all the coadjoint orbits of K can be realized as reduced spaces

$$\mathcal{O}_\lambda = (G /\!\!/ N) /\!\!/_\lambda T$$

for a singular affine variety $G /\!\!/ N = \mathrm{Spec}(\mathbb{C}[G]^N)$, equipped with a certain singular Kähler structure.[1]

(Fix an embedding of $G /\!\!/ N$ into a complex inner product space E. Each smooth piece of $G /\!\!/ N$ has the restriction of the Kähler structure on E)

Other interesting families of symplectic manifolds also appear this way:

- toric symplectic manifolds
- multiplicity spaces $\mathcal{O}_\lambda \times \mathcal{O}_\nu \times \mathcal{O}_\xi /\!\!/_0 K$

[1] This is a theorem of Guillemin-Jeffrey-Sjamaar

Question: Given an affine variety X with a singular Kähler structure, can we construct a continuous map (using toric degeneration techniques)

$$\mu\colon X \to \mathbb{R}^N$$

which restricts to the moment map of a completely integrable torus action on each smooth piece of X?

Question: Given an affine variety X with a singular Kähler structure, can we construct a continuous map (using toric degeneration techniques)

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which restricts to the moment map of a completely integrable torus action on each smooth piece of X?

Answer: Yes! Under certain reasonable conditions.

Naïve approach:

Find a toric degeneration $\pi \colon \mathcal{X} \to \mathbb{C}$ of X to an affine toric variety X_S.

The stratification of X into smooth pieces gives a stratification of $\mathcal{X}$ (away from zero fiber)

Kähler structure on X
$\rightsquigarrow$ Kähler structure on $\mathcal{X}$.

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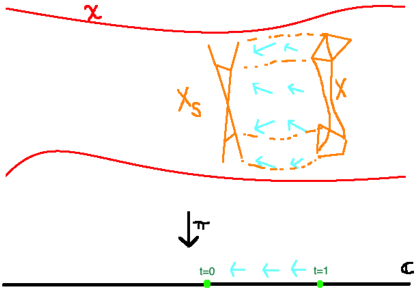

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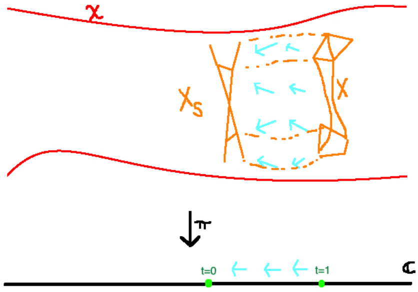

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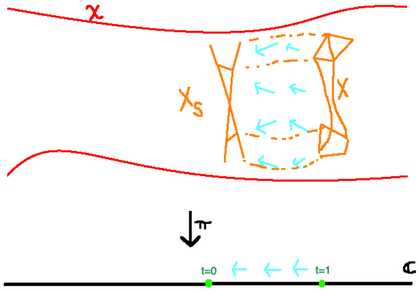

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Problem 1: The smooth pieces of X aren't compact.

Problem 2: Maybe the flows don't patch together nicely.

Let $A = \mathbb{C}[X]$, and $v\colon A\backslash\{0\} \to (\mathbb{Z}^N, <)$ a valuation with one dimensional leaves.

the ordering $<$ should be something reasonable
$v(fg) = v(f) + v(g)$ and $v(f+g) \leq \min\{v(f), v(g)\}$ and $v(\mathbb{C}^\times) = 0$
$\{f \mid v(f) \leq x\}/\{f \mid v(f) < x\}$ is zero- or one-dimensional for $x \in \mathbb{Z}^N$

Let $\mathsf{S} = v(A\backslash\{0\})$, and assume it is finitely generated.

Rees algebra construction: there is a toric degeneration $\pi\colon \mathcal{X} \to \mathbb{C}$ of X to X_{S}.

Let H be an algebraic torus. We require a linear "control map"

$$\mathsf{c}\colon \mathsf{S} \to X^*(H).$$

We additionally require:

- S is strictly convex, and $\mathsf{c}^{-1}(0) = \{0\}$.
- $\mathsf{c} \circ v\colon A\backslash\{0\} \to X^*(H)$ makes A into a $X^*(H)$-graded algebra
- The decomposition of X by H-orbit types is a Whitney A stratification into smooth manifolds
- The symplectic volume of $\pi^{-1}(1) /\!\!/_\lambda H$ is equal to symplectic volume of $\pi^{-1}(0) /\!\!/_\lambda H$, for $\lambda \in X^*(H) \otimes \mathbb{R}$

Theorem (H-Lane)

If there exists c *as above, there exists a continuous map $\mu\colon X \to \mathbb{R}^N$ which restricts to the moment map of a completely integrable torus action on each smooth stratum of X. And, $\mu(X) = \mathrm{cone}(\mathsf{S})$.*

Let K be any compact Lie group, and let λ be any dominant weight of K.

Theorem (H-Lane)

There is a completely integrable torus action on $(\mathcal{O}_\lambda, \omega_\lambda)$.

$+$ symplectic contraction arguments:

Let (M, ω, μ) be any compact Hamiltonian K-manifold, and assume $M /\!\!/_{\lambda} K$ is 0-dimensional for all $\lambda \in \mathfrak{t}^*$.

Theorem (H-Lane)

There is a completely integrable torus action on (M, ω).

Notably, some of these M are not Kähler!!