Infinite dimensional localization

&

Virasoro characters

joint with S. Shatashvili

1. Intro : $G = SU(2)$

2. Virasoro coadjoint orbits and formal DH integrals

3. Virasoro characters

1. $G = SU(2)$

Character of irrep of
$\dim = \lambda + 1$

$$\chi_\lambda(\theta) = \frac{\sin((\lambda+1)\theta)}{\sin\theta}$$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Highest weight $= \lambda$

Depth of singular vector $= \lambda + 1$

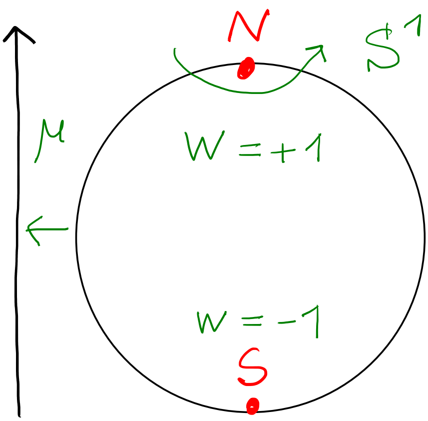
- <u>Scaling</u> :

$$\lambda \longmapsto k\lambda, \quad \theta \longmapsto \frac{\theta}{k}$$

$$\chi_{k\lambda}\left(\frac{\theta}{k}\right) \sim \frac{k}{2\pi}\cdot \underbrace{\frac{2\pi}{i\theta}\left[\frac{e^{i\lambda\theta}}{(+1)} + \frac{e^{-i\lambda\theta}}{(-1)}\right]}_{=\, I^{DH}_{\lambda}(\theta)}$$

DH integral:

$$I^{DH}_{\lambda}(\theta) = \int_{S^2_\lambda} \omega_\lambda \, e^{i\mu\theta}$$



S^2_λ = coadjoint orbit of $SU(2)$

$$I^{DH}_{\lambda}(\theta) = \lim_{k\to\infty} \frac{2\pi}{k}\, \chi_{k\lambda}\left(\frac{\theta}{k}\right)$$

2. Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} \delta_{n+m,0} (m^3 - m)$$

- $sl(2)$ subalgebras

$$sl(2)_m = \mathbb{C}\left\langle E_m = \frac{L_m}{m}, F_m = \frac{L_{-m}}{m}, H_m = \frac{L_0}{m} + \frac{c(m^2-1)}{24} \right\rangle$$

- Coadjoint action:

$$\mathrm{Diff}(S^1) \circlearrowright \{ b(s)\, ds^2 ;\ b(s+2\pi) = b(s) \}$$

$\cup$ quadratic differentials on S^1

S^1

rotations of S^1

- Coadjoint action:

$$f : b(s) \longmapsto b(f(s))\, f'(s)^2 + \frac{c}{12} S(f)$$

- Schwarzian derivative

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

- Moment map for S^1 action:

$$\mu(b) = \int_0^{2\pi} b(s)\, ds$$

Prop (Lazutkin - Pankratova, Witten, ...)

Assume $b = \text{const} = b_0$

$\Rightarrow$ • $\mathcal{O}_{b_0} \cong \mathrm{Diff}(S^1)/SL(2,\mathbb{R})$

if $b_0 = \dfrac{c\,m^2}{24}$, $m \in \mathbb{Z}_{>0}$

$\cong \mathrm{Diff}(S^1)/S^1$

otherwise

• $\exists!$ S^1 fixed point

$$\boxed{b(s) = b_0}$$

Remark: $\mu(b_0) = 2\pi b_0$

$$T_{b_0}\mathcal{O} \cong \left\{ f(s) \,;\, \int_{S^1} f(s)\,ds = 0 \right\}$$

$$\circlearrowleft$$

$$S^1$$

Formal DH integrals (Stanford-Witten)

$$\boxed{I^{DH}_{b_0}(t) = \frac{e^{\mu(b_0)t}}{Pf_S(A_t)}}$$

$t\,\partial_s$

action on $T_{b_0}\Theta$

- Stabilizer $= S^1$

$$I^{DH}_{b_0} = \left(\frac{t}{2\pi}\right)^{1/2} e^{2\pi b_0 t}$$

- Stabilizer $= SL(2,\mathbb{R})_m$

$$I^{DH}_{m} = m\left(\frac{t}{2\pi}\right)^{3/2} e^{\frac{\pi m^2 c}{12}t}$$

3. Virasoro characters

$$\chi_{(c,\Delta)} = \mathrm{Tr}_V \, q^{L_0 - \frac{c}{24}}$$

central charge — lowest weight

$$q = e^{-2\pi t}$$

- Irreducible Verma modules

$$\chi_{c,\Delta}(t) = \frac{q^{\Delta - \frac{c}{24}}}{\prod_{n=1}^{\infty} (1 - q^n)}$$

Prop :

$$\chi_{(kc,k\Delta)}\left(\frac{t}{k}\right) \underset{k\to\infty}{\sim} \underbrace{\left(\frac{2\pi}{k}\right)^{1/2} e^{\frac{\pi k}{12t}}}_{\text{divergent factor}} \, \mathcal{I}^{DH}_{-(\Delta - \frac{c}{24})}(t)$$

- Degenerate Verma modules

$$c = 1 + 6\left(\beta + \beta^{-1}\right)^2$$

$$\Delta = \frac{1}{4}\left(\left(\beta + \beta^{-1}\right)^2 - \left(r\beta + s\beta^{-1}\right)^2\right)$$

$\Rightarrow$ $\exists$ a singular vector in depth

$$m = rs$$

$$\chi_{(c,\Delta)}(t) = \frac{q^{\Delta - \frac{c}{24}}\left(1 - q^m\right)}{\prod_{n=1}^{\infty}\left(1 - q^n\right)}$$

Question: scaling behavior?

__Scaling 1 :__ $\beta \mapsto k^{-1/2}\beta$, $t \mapsto k^{-1} t$

$r, s = \text{const}$

$$\Rightarrow \quad \chi_{(c(k),\Delta(k))}\left(\frac{t}{k}\right) \underset{k\to\infty}{\sim} \left(\frac{2\pi}{k}\right)^{3/2} e^{\frac{\pi k}{12 t}} \cdot r\, I_s^{DH}(t)$$

for $c = 6\beta^{-2}$

__Scaling 2 :__ $\beta \mapsto k^{-1/2}\beta$, $t \mapsto k^{-1} t$,

$r \mapsto k\, r$, $s \mapsto s$

$$\Rightarrow \quad \chi_{(c(k),\Delta(k))}\left(\frac{t}{k}\right) \underset{k\to\infty}{\sim} \left(\frac{2\pi}{k}\right)^{1/2} e^{\frac{\pi k}{12 t}} \cdot \left(\frac{t}{2\pi}\right)^{1/2} e^{2\pi b_0 t}\left(1 - e^{-2\pi r s t}\right)$$

$$b_0 = \frac{1}{4}\left(r^2\beta^2 + 2rs + s^2\beta^{-2}\right)$$

no such coadjoint orbits!

Thank you !

More :

- ∞ many singular vectors
- volume functions
- spectral presentation

∞ many singular vectors :

$$c = 1 + 6(\beta + \beta^{-1})^2$$

$$\Delta = \frac{1}{4}\left((\beta+\beta^{-1})^2 - (r\beta + s\beta^{-1})^2\right)$$

- $\beta^2 = \frac{p}{q}$, $1 \leq r \leq q$, $1 \leq s \leq p$

  $P_\pm = r\beta \pm s\beta^{-1}$

- scaling: $\beta \mapsto k^{-1/2}\beta$, $p \mapsto p$, $s \mapsto s$

  $q \mapsto kq$, $r \mapsto kr$

- character:

$$\chi_{kr,s}\left(\frac{t}{k}\right) \underset{k\to\infty}{\sim} \left(\frac{2\pi}{k}\right)^{\frac{1}{2}} e^{\frac{\pi k}{12t}} \cdot$$

$$\cdot \left(\frac{t}{2\pi}\right)^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} \left(e^{-2\pi(P_+ + n\sqrt{pq})^2 t} - e^{-2\pi(P_- + n\sqrt{pq})^2 t}\right)$$

∞ number of fixed points ?

Volume functions

- $G = SU(2)$

$$V(x) = \frac{1}{2\pi}\int_{\mathbb{R}+i\varepsilon} \underbrace{\frac{e^{i\lambda\theta} - e^{-i\lambda\theta}}{i\theta}}_{I^{DH}_{\lambda}(\theta)} e^{-ix\theta}\, d\theta = \begin{cases} 1 & \text{if } |x| < r \\ 0 & \text{otherwise} \end{cases}$$

- Virasoro

$$V(x) = \int \left(\frac{t}{2\pi}\right)^{1/2} e^{(2\pi b_0 - x)t}\, dt = \begin{cases} 0 & \text{if } x > 2\pi b_0 \\ \dfrac{C}{(2\pi b_0 - x)^{3/2}} & \text{otherwise} \end{cases}$$

volume of ∞-dim reduced space ?

Spectral representation

(Saad – Shenker – Stanford)

- $$I^{DH}\left(\frac{1}{\beta}\right) = \int_0^\infty \rho(E)\, e^{-\beta E}\, dE$$

- $$\rho(E) = \frac{1}{2\pi} \int I^{DH}\left(\frac{1}{\beta}\right) e^{\beta E}\, d\beta$$

E.g.

$$\rho_m(E) = C \sinh\left(2m \left(\frac{\pi C}{12} E\right)^{1/2}\right)$$

for $E \geq 0$

link to matrix models