Deformations of Poisson structures on Hilbert schemes

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Based on joint work in progress with

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Holonomicity: nondegeneracy condition for holomorphic Poisson structures (P.–Schedler)

$$(X,\pi) \quad \rightsquigarrow \quad (\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi) \otimes \mathcal{D}_X \quad \rightsquigarrow \quad \mathit{Char}(X,\pi) \subset T^*X$$

$$\text{Holonomic} \iff \mathit{Char}(X,\pi) \text{ Lagrangian} \underbrace{\Longrightarrow}_{\mathit{conj} \iff} \# \text{ char leaves} < \infty$$

Symplectic leaf is **characteristic** if modular vector field $\Delta\pi$ is tangent

Motivation: $(\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi)$ is perverse, so deformation theory is "topological"

This talk: an illustrative example

(X,π) smooth Poisson surface $\quad \rightsquigarrow \quad$ $\mathit{Hilb}^n(X,\pi)$ its Hilbert scheme

Poisson surface := $\mathbb{C}$-surface X with $\pi \in H^0(K_X^{-1})$

$\partial X := Zeros(\pi) \subset X$

$\partial^2 X :=$ singular locus of ∂X

Nondegenerate on $X^\circ := X \setminus \partial X$:

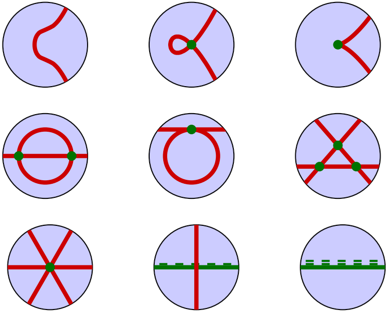
$\pi \cong \partial_q \wedge \partial_p \qquad \Delta\pi = 0$

On smooth locus of ∂X:

$\pi \cong u\partial_u \wedge \partial_v \qquad \Delta\pi = \partial_v$

Characteristic leaves: X°, $\partial^2 X$

$X = \mathbb{P}^2 \quad deg(K_X^{-1}) = 3 \quad Y =$ cubic



holonomic $\iff$ ∂X reduced $\iff$ $\omega := \pi^{-1}$ log symplectic

Theorem (Enriques, Kodaira; Bartocci–Macrí, Ingalls)

If (X, π) is a projective Poisson surface, then (X, π) is birational to:

$(\mathbb{P}^2, cubic) \qquad T^*(curve) \qquad (\mathbb{P}^1 \times \frac{\mathbb{C}}{\Lambda}, u\partial_u \wedge \partial_v) \qquad (\frac{\mathbb{C}^2}{\Lambda}, \partial_q \wedge \partial_p) \qquad K3$

Consequently, ∂X is locally quasi-homogeneous.

Characteristic leaves: $X^\circ \xhookrightarrow{j} X \xhookleftarrow{i} \partial^2 X$

Theorem (Goto, P.–Schedler)

$\partial^2 X$ quasi-homogeneous $\implies (\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi) \cong Rj_* \mathbb{C}_{X_0} \oplus i_* i^* K_X^{-1}[-2]$, *so*

$$HP^j(X, \pi) \cong \begin{cases} H^j(X^\circ; \mathbb{C}) & j \neq 2 \\ \underbrace{H^2(X^\circ; \mathbb{C})}_{\text{defs. of } \omega} \oplus \underbrace{H^0(i^* K_X^{-1})}_{\text{smoothings of } \partial^2 X} & j = 2 \end{cases}$$

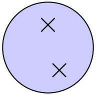
Sketch of proof.

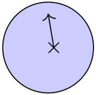
1. Restriction to open leaf: $j^*(\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi) \cong (\Omega^\bullet_{X^\circ}, d) \cong \mathbb{C}_{X^\circ}$
2. Therefore (adjunction): $\wedge^\bullet \mathcal{T}_X \to Rj_* \mathbb{C}_{X^\circ}$
3. Splitting: $Rj_* \mathbb{C}_{X^\circ} \cong \Omega^\bullet_X(\log \partial X) \to \wedge^\bullet \mathcal{T}_X$ via quasihomogeneous "log comparision" of [Castro-Jiménez–Narváez-Macarro–Mond]
4. Show that $\wedge^\bullet \mathcal{T}_X / \Omega^\bullet_X(\log \partial X) \cong i_* i^* K_X^{-1}[-2]$

$$\underbrace{Sym^n(X) := X^n/S_n}_{\text{singular Poisson variety}} \longleftarrow \underbrace{Hilb^n(X) := \{\text{length-}n \text{ subschemes of } X\}}_{\text{smooth Poisson [Beauville, Bottacin, Mukai]}}$$

For instance:

$$Hilb^2(X) = Bl_\Delta Sym^2(X) = \underbrace{(Sym^2(X) \setminus \Delta)}_{\text{reduced schemes}} \sqcup \underbrace{\mathbb{P}(T_X)}_{\text{1-jets}}$$





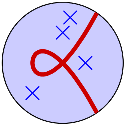
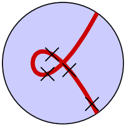
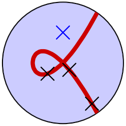
Case X compact Kähler, π nondegenerate:

- Same for $Hilb^n(X)$ [Beauville, Mukai], so hyperKähler [Calabi, Yau]
- Albanese fibres are "irreducible" [Beauville] – only other known IHSMs (up to deformation) are O'Grady's M^6, M^{10}
- Unobstructed deformations parameterized by $H^2(Hilb^n(X); \mathbb{C})$ [Beauville, Bogomolov]

Symplectic leaves of $Hilb^n(X)$

$$W \in Hilb^n(X) \quad \rightsquigarrow \quad \partial W := W \cap \partial X \quad \text{(scheme-theoretic)}$$

$$W, W' \text{ in same symplectic leaf} \iff \partial W = \partial W'$$

∂W	Example	Leaf
$\varnothing$		$Hilb^n(X^\circ)$
W		$\{W\}$
$n-1$ points		$(Bl_{\partial W} X)^\circ$

Characteristic leaves

Locally: modular vector field $\Delta\pi$ on X lifts to $\Delta\pi_{Hilb}$

Proposition

leaf of W is characteristic (i.e. $\Delta\pi_{Hilb}$ tangent) $\iff$ ∂W fixed by $\Delta\pi$

Conjecture (Matviichuk–P.–Schedler)

For $n \geq 2$, we have:

$$\begin{aligned} Hilb^n(X) \text{ holonomic} &\iff \#\ \text{char leaves} < \infty \\ &\iff \text{every point in } X \text{ has type } A_k, k \geq 0 \\ &\qquad \text{i.e. local equation } x^2 = y^{k+1} \end{aligned}$$

Cases proven so far:

- both $\implies$, both
- both $\impliedby$ for $n = 2$ or ∂X smooth
- second $\impliedby$ for $k \leq 2$

Key point: type A_k $\iff$ linearization of $\Delta\pi$ nonzero

Theorem (Matviichuk–P.–Schedler)

For (X,π) connected log sympl. surf., ∂X locally quasi-hgns, $n \geq 2$:

$$\begin{aligned} HP^2(Hilb^n(X)) &= H^2(Hilb^n(X^\circ)) \oplus H^0(i^* K_X^{-1}) \\ &= \underbrace{HP^2(X)}_{Hilb(Def(X,\pi))} \oplus \underbrace{\wedge^2 H^1(X^\circ;\mathbb{C})}_{log\ \ Albanese} \oplus \underbrace{\mathbb{C}\cdot c_1(E)}_{Hilb(Quant(X,\pi))} \end{aligned}$$

[Hitchin, Nevins–Stafford, Rains]

Corollary (Ran)

If ∂X is smooth then deformations are unobstructed.

Corollary

Rains' Hilbert schemes of noncommutative rational surfaces form irreducible components in the moduli space of Poisson varieties

Deformations – proof

Theorem (Matviichuk–P.–Schedler)

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[Hitchin, Nevins–Stafford, Rains]

Sketch.

1. throw out codim 4 (higher Hartogs); look at char. leaves
2. codim 0: $Rj_*\mathbb{C}_{Hilb^n X^\circ}$, codim 2: $Rj'_*\mathbb{C}_{Hilb^{n-1}X^\circ} \otimes H^0(i^* K_X^{-1})$
3. codim 1: no contributions, codim 3: could a priori only make HP^2 smaller, but doesn't (local calculation, or interpret deformations) □

THANK YOU!