Talk: Deformations of toric Poisson structures

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Based on joint work with Pym, Matchiivuk–Pym

I. Motivation + setting

II. Holonomic Poisson structures (or log symplectic)

III. Normal crossings case (eg toric)

IV. Examples ($\mathbb{P}^2$, $\mathbb{C}^4$, $\mathbb{P}^4$)

Slides are on my Imperial page ("Talks and Lectures"):

https://www.imperial.ac.uk/people/t.schedler/page/talks-and-lectures.html

I. Motivation and setting

Goal: To understand moduli of holomorphic Poisson manifolds (or Poisson $\mathbb{C}$-alg. varieties).

↳ gives new examples of Poisson structures

↳ elucidates ones we know, classifies their deformations

↳ Also applies to symplectic / log symplectic manifolds

Example: Harder preprint on toric Poisson degenerations of hyperkähler manifolds

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Moduli space: $\mathcal{M} := \{ (X, \pi)$ holomorphic Poisson manifold (or $\mathbb{C}$ alg. variety) $\}$

· $\pi \in \Gamma(X, \Lambda^2 T_X)$ Poisson bivector (manifold case).

Difficult to understand.

E.g.:
- Fixing $X = \mathbb{P}^3$, Cerveau–Lins Neto proved: $\mathcal{M}$ has 6 irreducible components.
- Using techniques from this project, we find ~ 40 (new) components for $X = \mathbb{P}^4$.

⚠ In general, X can also vary in $\mathcal{M}$; but, there are no deformations of $X=\mathbb{P}^n$.

Local approach:

$$T_{(X,\pi)}\mathcal{M} \cong HP^2(X,\pi)$$

$\uparrow$ 2nd Poisson cohomology.

Formal neighbourhood:

Governed by dgla (via Maurer–Cartan formalism)

differential graded Lie algebra

$$\mathcal{L}^\bullet_{X,\pi} := \left(\Lambda^\bullet T_X,\ d_\pi := [\pi, -]_{\text{Schouten-Nijenhuis}}\right) =$$ $\underline{\text{Sheaf}}$ of dglas (shifted)

(BACKGROUND: Schouten-Nijenhuis bracket extends Lie bracket $\Lambda^1 T_X = T_X \times T_X \longrightarrow T_X$ so as to be a (graded) derivation: $[\xi, \eta\wedge\eta'] = [\xi,\eta]\wedge\eta' + (-1)^{|\eta||\eta'|}[\xi,\eta']\wedge\eta$.)

Reason:

A formal deformation of π:

$$\pi' = \pi + \hbar\pi_1 + \hbar^2\pi_2 + \dots$$

Jacobi identity $\Longleftrightarrow$ $\cdot\ [\pi,\pi_1] = 0 \iff \pi_1 \in \mathcal{L}^2_{X,\pi}$ d_π-$\underline{\text{closed}}$

$[\pi',\pi']=0$ $\quad \cdot\ \frac{1}{2}[\pi_1,\pi_1] + [\pi,\pi_2] = 0 \iff [\pi_1,\pi_1] \subset \mathcal{L}^3_{X,\pi}$ $\underline{\text{coboundary}}$

$\vdots$

Rewritten: $\pi'' := \pi' - \pi = \hbar\pi_1 + \hbar^2\pi_2 + \dots$ satisfies

Maurer-Cartan equation: $\boxed{\frac{1}{2}[\pi'',\pi''] + d_\pi(\pi'') = 0.}$

⤳ Deformations of π (leaving X fixed)

$\updownarrow$

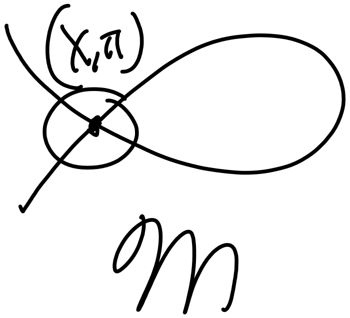
Maurer-Cartan elements of $\Gamma(\mathcal{L}^{\bullet-1}_{X,\pi})$.

shift to get a dgla

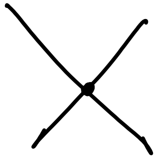
Allowing X to vary (+ including "twisted Poisson deformations"):

Formal deformations of X, π ($\widehat{\mathcal{M}}_{X,\pi}$ = formal neighbourhood) $\longleftrightarrow$ Maurer-Cartan elements of $\mathfrak{g} := R\Gamma(\mathcal{L}^{\bullet-1}_{X,\pi})$ = some big dgla (resolve sheaf $\mathcal{L}^{\bullet}_{X,\pi}$)

Toy picture:



⤳



Maurer-Cartan elements

$$\widehat{\mathcal{M}}_{X,\pi} = MC(\mathfrak{g}) \big/ \text{gauge equivalence}$$

Examples:

- $\pi = 0$: $\mathcal{L}_{X,\pi} = \Lambda^{\bullet} T_X$, zero differential

See ALL Poisson structures on X,
ALL deformations of X: COMPLICATED.

HIGHLY DEGENERATE

NON-DEGENERATE

- (X,π) symplectic: $\omega \in \Omega^2(X)$ $d\omega = 0$, $\begin{tikzcd} T_X^* \arrow[r, shift left, "\iota_\pi"] \arrow[r, phantom, "\cong"] & T_X \arrow[l, shift left, "\omega^\#"] \end{tikzcd}$

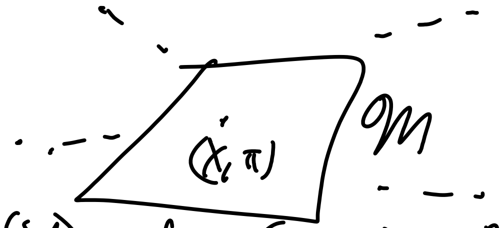
$(\omega^\#(\xi) = i_\xi \omega)$

$\Rightarrow \mathcal{L}_{X,\pi} \cong (\Omega^{\bullet}_X, d_{DR}) \simeq \mathbb{C}_X$, $HP^{\bullet}(X) \cong H^{\bullet}(X) \simeq \mathfrak{g}$

abelian dgla with zero differential.

$\Rightarrow$ deformations "unobstructed",

i.e. $\widehat{\mathcal{M}}_{X,\pi} \cong \widehat{HP^2(X)} \overset{\text{here}}{\cong} \widehat{H^2(X)}$



$\Leftrightarrow \mathcal{M}_{X,\pi}$ is a manifold of dim $H^2(X)$ at $(X,\pi) \in \mathcal{M}$.

"EASY" (up to identifying the deformations).

II. Holonomic Poisson structures

Moral: Deformation theory is easier / more controllable the more non-degenerate (X,π) is.

Goto: A log symplectic structure is a closed meromorphic two-form $\omega \in \Gamma(X, \Omega^2(\log D))$, $D \subseteq X$ divisor = hypersurface (reduced)

s.t.

$$\begin{tikzcd} \Omega^1_X(\log D) \arrow[r, shift left, "i_\pi"] \arrow[r, phantom, "\cong"] & T_X(-\log D) \arrow[l, shift left, "\omega^\#"] \end{tikzcd}$$

$\Rightarrow (X,\pi)$ Poisson

Example: $X = \mathbb{P}^n$, $D = \mathbb{P}^{n-1}_0 \cup \dots \cup \mathbb{P}^{n-1}_k$; $k \le n$

$$\Rightarrow \Omega^1_X(\log D) = \mathrm{Span}_{\mathcal{O}_X}\left(\frac{dx_i}{x_i} \,\Big|\, i \le k,\ dx_i \,\Big|\, i > k\right)$$

$$T_X(\log D) = \mathrm{Span}_{\mathcal{O}_X}\left(x_i \partial_i \mid i \le k,\ \partial_i \mid i > k\right)$$

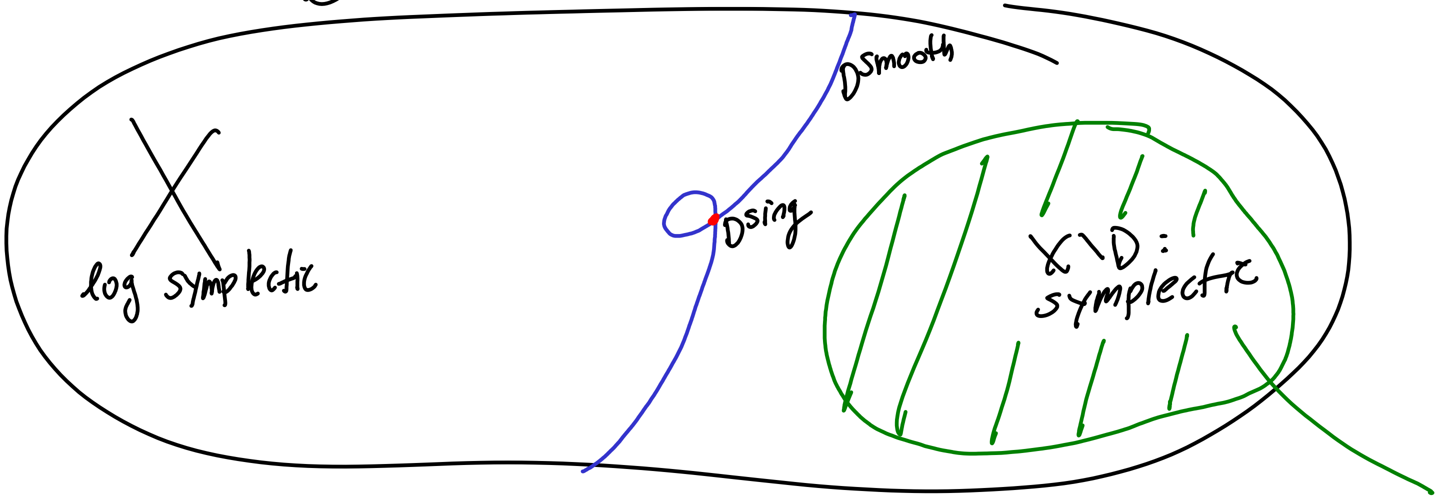
E.g.: $\omega = \sum_{i=1}^{n/2} \frac{dp_i}{p_i} \wedge dq_i$, $\pi = \sum_{i=1}^{n/2} p_i \partial_{p_i} \wedge \partial_{q_i}$, $D = \bigcup_{i=1}^{N} \{p_i = 0\}$.

Proposition (Goto): (X,π) log symplectic $\Leftrightarrow$ $D :=$ degeneracy locus is $\boxed{\text{reduced}}$.

$$D = Z\left(\mathrm{Pf}(\pi) = \Lambda^{\dim X/2}\pi\right)$$

↖ section of anticanonical bundle

Proposition: In this case, $\mathcal{L}_{(X,\pi)}$ is locally finite-rank on D^{smooth}: $\mathcal{L}_{(X,\pi)}\big|_{X\setminus D^{sing}} \simeq \left(\Omega^\bullet_X(\log D)\big|_{X\setminus D^{sing}}, d_{DR}\right)$



$$\mathcal{L}_{X\setminus D} \simeq \Omega^\bullet_{X\setminus D} \simeq \mathbb{C}_{X\setminus D}$$

Hierarchy: (X, π) is:

- Generically symplectic $(D = Z(\mathrm{Pf}\,\pi) \subsetneq X)$ $\iff$ $\mathcal{L}_{X,\pi}$ finite-rank outside codim ≥ 1 $\Rightarrow$ $HP^0(X) = \mathbb{C}$
  D
  finite-dimensional
- log symplectic (D reduced) $\iff$ $\mathcal{L}_{X,\pi}$ finite-rank outside codim ≥ 2 $\Rightarrow$ $HP^1(X) = \langle \xi_{\log g} \rangle_{Z(g) \subseteq D}$
  D^{sing}
  finite-dimensional

$\vdots$

Defn (Pym–S.) (X, π) is **holonomic** if

D-module version $M_{X,\pi} := \mathrm{Diff}(\mathcal{O}_X, \mathcal{L}_{X,\pi})$ has **holonomic cohomology**

ie, Lagrangian support $\subseteq T^*X$

$\xrightarrow{\text{Kashiwara}}$ $\mathcal{L}_{X,\pi}$ **constructible** $\Rightarrow$ $\mathcal{L}_{X,\pi}$ locally of finite rank

Prop (Pym–S.) In this case, $\mathcal{H}^i(M_{X,\pi}) = 0$, $i \neq 0$

$\xrightarrow{\text{Kashiwara}}$ $\mathcal{L}_{X,\pi}[\dim X]$ **perverse**.

In particular, X is <u>stratified</u>,

$\mathcal{L}_{X,\pi}$ built out of local systems $L_i[-d_i]$ on strata of codim d_i.

$X \supsetneq D \supsetneq D^{\mathrm{sing}} \supsetneq \cdots$ (refines iterated singular loci)

Prop (Pym–S.) This includes symplectic leaves $Z \subseteq X$ such that $\xi_{\log g} = \Delta_\pi$ tangent to Z ($\{g=0\}=D$ locally)

Defn: $\boxed{\text{"characteristic symplectic leaves"}}$

Conjecture (Matviichuk–Pym–S.)

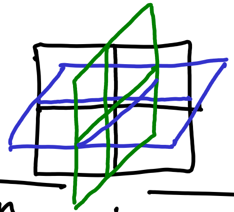
X holonomic $\Longleftrightarrow$ # characteristic symplectic leaves $< \infty$.

By Prop, $\Longrightarrow$ holds.

Theorem (Matviichuk–Pym–S.)

Conjecture holds if D has <u>normal crossings</u>

i.e. $\forall z \in D$, $\widehat{D_z} \cong \{x_1 x_2 \cdots x_k = 0\} \subseteq \widehat{\mathbb{C}^n}$
$n = \dim X$



$n=k=3$: $\{xyz=0\} \subseteq \widehat{\mathbb{C}^3}$

E.g. toric Poisson $\Rightarrow$ normal crossings, $X \setminus D = (\mathbb{C}^\times)^n$, strata are tori.

III. Normal crossings case (eg toric)

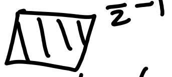
(10/17)

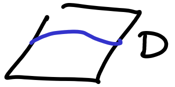
$$X \supseteq D =: D^1 \supseteq D^2 \supseteq D^3 \supseteq \dots$$

$D^k :=$ D locally $\cong \{x_1 \cdots x_k = 0\}$.

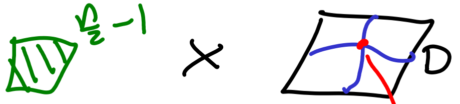
Question: is there a log symplectic manifold nonholonomic in codim 2?

- $D \setminus D^1$: symplectic


$\frac{n}{2}-1$

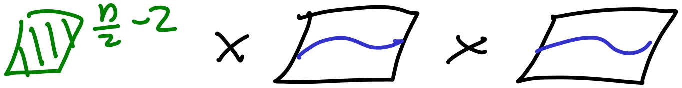


- At $x \in D^1 \setminus D^2$: a neighbourhood $\cong \underset{\text{dim } n-2}{\text{symplectic}} \times$ log $\underset{\text{dim } 2}{\text{symplectic surface}}$
- At $x \in D^2 \setminus D^3$ [Ran] : Either: (a) As above (



), OR

(b)



Characteristic symplectic leaf

$\Rightarrow$ holonomic outside D^3.

Proposition (Matchiivuk–Pym–Schedler):

- $HP^{\leq 2}(X) \cong H^{\leq 2}(\underset{\text{symplectic}}{X \setminus D}) \oplus \underset{\substack{Z \text{ characteristic} \\ \text{codim 2 leaf} \\ (▨^{\frac{n}{2}-1} \times \bullet \text{ in (a)})}}{\bigoplus} H^0(\overline{Z}, \mathcal{L}_Z^{nr})[-2]$

  $\mathcal{L}_Z^{nr}$: local system in $\mathcal{L}_{X,\pi}$ ("nonresonantly" extended to $\overline{Z}$)

  $$H^0(\overline{Z}, \mathcal{L}_Z^{nr}) = \begin{cases} \mathbb{C}, & \text{if } \mathcal{L}_Z \text{ trivial, "nonresonant" "Z smoothable"} \\ 0, & \text{if } \mathcal{L}_Z \text{ nontrivial or resonant} \end{cases}$$

- If $X \setminus D^4$ holonomic,

  $HP^{\leq 3}(X) \cong H^{\leq 3}(X \setminus D) \oplus \bigoplus H^{\leq 1}(Z, \mathcal{L}_Z)[-2]$

Description of ω and leaves Z:

Let $D = D_1 \cup \dots \cup D_m$ = irreducible components.

(assume wlog simple normal crossings: D_i do not self intersect)

$\Rightarrow D^p = \bigcup D_{i_1 \dots i_p} := D_{i_1} \cap \dots \cap D_{i_p}$. $\quad \overset{\circ}{D}_{i_1 \dots i_p} := D_{i_1 \dots i_p} \setminus D^{p+1}$.

Defn $\mathrm{Bires}_{i,j}\, \omega := \mathrm{Res}_{D_i} \mathrm{Res}_{D_j}\, \omega \in \underbrace{\Gamma(\overset{\circ}{D}_{ij}, \mathbb{C})}_{\text{locally constant}}$

$$\|$$

$$- \mathrm{Res}_{D_j} \mathrm{Res}_{D_i}\, \omega$$

$$\|$$

$$\frac{1}{(2\pi i)^2} \int_{\Sigma_{ij}} \omega \, , \quad \Sigma_{ij} = \text{torus wrapping } D_{ij} \text{ at a point.}$$

Prop Up to taking products with symplectic spaces (stable equivalence),

at $x \in D^m$, $\quad \omega \sim \sum_{i=1}^{m} \frac{dp_i}{p_i} \wedge dq_i + \sum_{i<j} \mathrm{Bires}_{ij}\, \omega(x) \frac{dq_i}{q_i} \wedge \frac{dq_j}{q_j}$.

Defn Given component $Z \subseteq D^{\circ}_{i_1 \cdots i_p}$ ("stratum"),

$$B_Z := \left(\mathrm{Bires}_{i_j i_k}\, \omega|_Z\right)_{1 \le j,k \le p}$$ skew-symmetric matrix.

∴ Determines π near Z up to stable equivalence.

Thm (MPS): (a) Corank (symplectic leaves in Z) = corank(B_Z)
(corank $\pi|_Z$)

(b) Symplectic leaves in Z are characteristic
$\iff (1, \dots, 1) \in \mathrm{im}(B_Z)$

Cor: (X, π) holonomic $\iff$ $\forall Z$ as in (b), Bires_Z is invertible,

i.e.: $\boxed{Z \text{ is a characteristic symplectic leaf.}}$

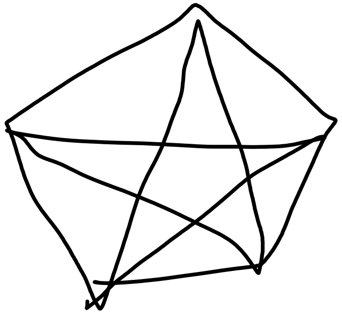
Thm In holonomic case, $\mathcal{L}_{X,\pi}$ admits a "weight filtration", subquotients are (sums of) extensions of rank one local systems L_Z on characteristic symplectic leaves (one L_Z for each Z).

Diagrams: Given (X, π) normal crossings, have

dual graph $\Sigma(X)$:
- vertices = components $D_i \subseteq D = D^1$
- k-cells = components of D^{k-1}

e.g.: $X = \mathbb{P}^4$, toric structure:

$$D = \mathbb{P}^3_0 \cup \cdots \cup \mathbb{P}^3_4 \ , \quad \mathbb{P}^3_i = \{x_i = 0\}$$

dual graph = , complete graph since $\mathbb{P}^3_{i_1} \cap \cdots \cap \mathbb{P}^3_{i_p} \cong \mathbb{P}^{4-p} \neq \phi$, **irreducible**.

Color an edge red if it corresponds to a **smoothable** stratum (characteristic leaf with trivial L_Z)

Prop: $Z \subseteq D^\circ_{ij}$ **smoothable** $\iff$ $\forall k \notin \{i,j\}$, $\underline{Y \subseteq \overline{Z} \cap D_k \text{ stratum}}$,

biresidues of ω along Y $= B_Y = \begin{array}{c|ccc} & i & j & k \\ \hline i & 0 & a & b \\ j & -a & 0 & c \\ k & -b & -c & 0 \end{array}$ for $\boxed{\frac{c-b}{a} \in \mathbb{Z}_{\geq 0}}$.

Color an angle red if it corresponds to a k
with $\frac{c-b}{a} \geq 1$, opposite a red edge.

<u>Thm (MPS):</u> • For (X,π) toric, dim n, $D = D_1 \cup \dots \cup D_m$

$$\widehat{\mathcal{M}}_{(X,\pi)} \cong \bigcup V_I \big/ (\mathbb{C}^\times)^n \rtimes \Gamma_I$$

smooth these strata → $I \subseteq \{$smoothable strata $=$ red edges in graph$\}$

$$V_I \cong \underbrace{\mathbb{C}^I}_{\text{smooth components in } I} \times \underbrace{H^2(X\setminus D)^I}_{\substack{\text{Deformations of } \omega \text{ leaving} \\ I \text{ smoothable}}}$$ <u>linear space</u> $\subseteq HP^2(X,\pi)$

$\Gamma_I = \{$permutations of codim two strata coming from $\mathrm{Aut}(X,\pi,I)\}$.

<u>To find components of $\mathcal{M}$:</u> suffices to let $I =$ all red edges

(up to moving π in $H^2(X\setminus D)^I$)

$\rightsquigarrow$ just count colorings of dual graph $/\cong$!

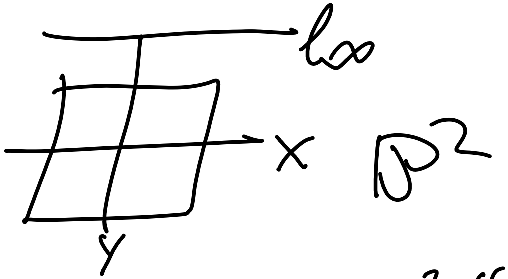
IV. Examples ($\mathbb{P}^2, \mathbb{C}^4, \mathbb{P}^4$)

Baby case: $\mathbb{P}^2$. $HP^3(\mathbb{P}^2) = 0$ since $\dim \mathbb{P}^2 = 2 < 3$

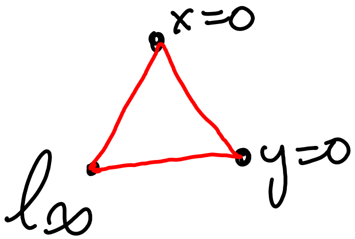
$\Rightarrow$ no obstructions; $\mathcal{M}_{\mathbb{P}^2}$ irreducible.

(In fact: $\mathcal{M}_{\mathbb{P}^2} = \{\pi = f(x,y)\,\partial_x \wedge \partial_y \mid \deg f \le 3\}/\cong$)

Toric structure: $\pi_0 = xy\,\partial_x \wedge \partial_y$.



Dual graph:



$$HP^2(X,\pi_0) = \underbrace{\mathbb{C}^3}_{\substack{\text{smooth}\\ \text{3 strata}}} \oplus \underbrace{\mathbb{C}}_{\substack{\text{rescaling}\\ \pi_0}} = H^2((\mathbb{C}^\times)^2)$$

$$\widehat{\mathcal{M}_{(\mathbb{P}^2,\pi_0)}} \cong \widehat{\mathbb{C}\times\mathbb{C}^3 / \underset{\substack{\uparrow\\ \text{torus}\\ \text{(dilate } x,y)}}{(\mathbb{C}^\times)^2} \rtimes A_3} \cong \widehat{\mathbb{C}^2}.$$

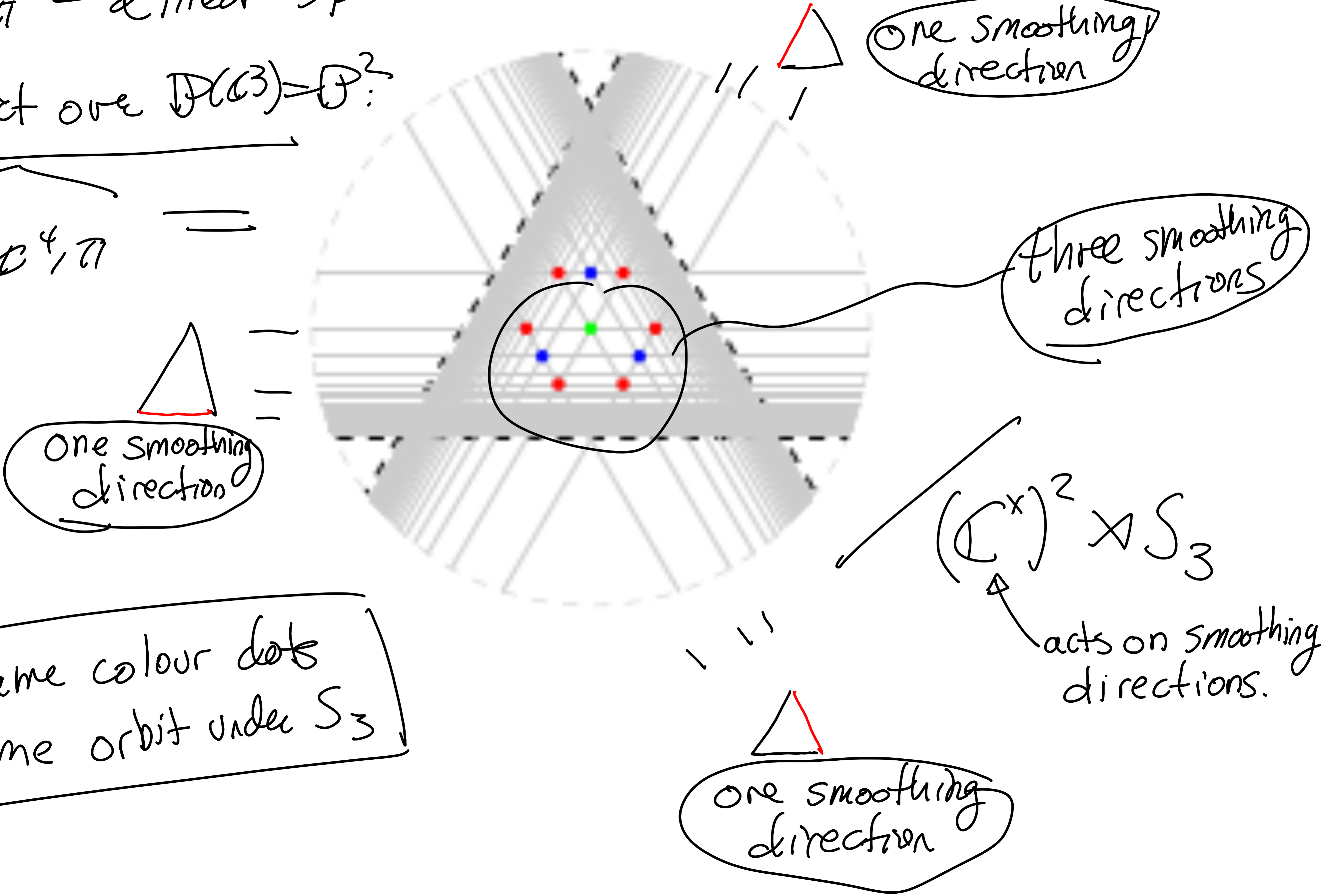
$\nwarrow$ cyclic permutations

$\mathbb{C}^4$, dual graph Δ ($D = \{x=0\} \cup \{y=0\} \cup \{z=0\}$):

$\widehat{\mathcal{M}_{\mathbb{C}^4,\pi}}$ = union of linear spaces over $\widehat{H^2(\mathbb{C}^4 \setminus D)} \cong \widehat{\mathbb{C}^3}$

Depict over $\mathbb{P}(\mathbb{C}^3) = \mathbb{P}^2$:

$\widehat{\mathcal{M}}_{\mathbb{C}^4,\pi} =$



$(\mathbb{C}^\times)^2 \rtimes S_3$

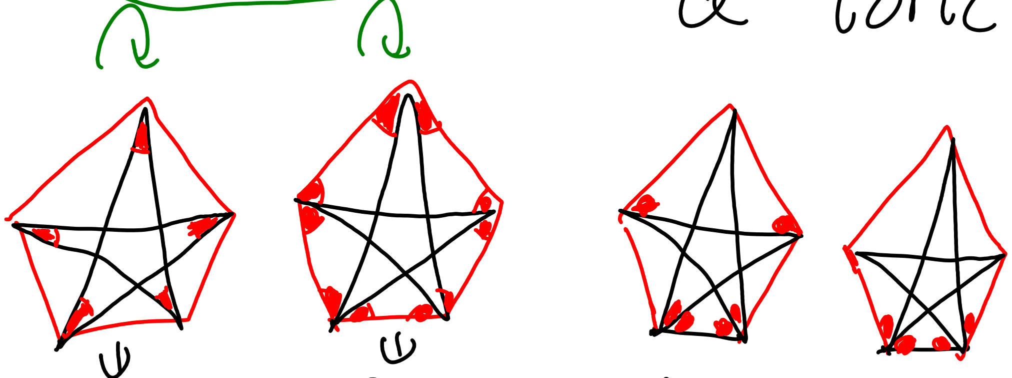
acts on smoothing directions.

Same colour dots = same orbit under S_3

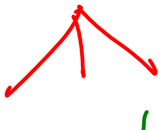
$\mathbb{P}^4$ case:

Thm (MPS): ~40 irreducible components of $\mathcal{M}$ for $X=\mathbb{P}^4$ containing a toric holonomic structure

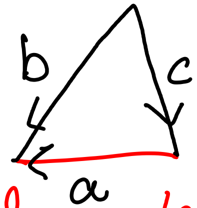
dihedral symmetry



(Feigin-Odesskii)

Rules: · No  (≤ 2 red edges at each vertex)

· Have to label edges by biresidues; given

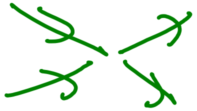


$\Rightarrow \frac{b+c}{a} \in \mathbb{Z}_{\geq 0}$

Red angle: ≥ 1

Graph determines geometry of smoothed D. (which components glued, lower strata...)

$\mathbb{P}^{2n}$ case:

- at most two red angles opposite each red edge
- Kirchoff law holds:  Σ biresidues in $=\Sigma$ biresidues out

...