May 28

Workshop on Polyhedral Products in GGT

Coxeter groups, Artin groups and buildings of type FC based on

"Examples of buildings constructed via covering spaces" GGD 3 (2009)

- Using polyhedral products to construct CAT(0) cube complexes for RACGs, RAAGs, RABs
- Analogous construction in non-right-angled case
- Main use: constructing new examples of buildings
- Many of these examples can be constructed (replaces their paper) using complexes of groups and theorem of Tits

Review of CAT(0) cube complexes for RACGs (and RAAGs and RABs)

L = flag cx, S = its vertex set

$[-1,1] = \mathrm{Cone}\{\pm 1\}$

$X_L = ([-1,1],\{\pm 1\})^L \subset \prod_{s \in S} [-1,1]$ = cube

$(C_2)^S \curvearrowright$ cube & X_L is invariant

(Kevin used Y_L instead of X_L)

Fundamental Chamber is

$K = ([0,1],1)^L = X_L \cap [0,1]^S$

$\tilde{X}_L$ = universal cover of X_L = cube cx

W_L = gp of all lifts of $(C_2)^S \curvearrowright X_L$ to $\tilde{X}_L$

= RACG associated to L^1

$\tilde{X}_L = \Sigma(W,S)$ = "Davis-Moussong cx"

Alternative description

$$\boxed{\Sigma(W,S) = D(W,K) = (W \times K)/\sim}$$

$\mathcal{U}(W,K)$ = basic construction

K_s = "panel"

<u>RAB</u>s:

rank 1 bldg = discrete set E_s

(bldg type $(C_2, \{s\})$)

its realization is Cone E_s

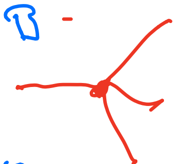
...

Product Blds $= \prod_{\Delta \in S} \operatorname{Cone} E_\Delta$ blds of type $((C_2)^S, S)$

As before $L =$ flag cx, vertex set S

$Z_L = (\operatorname{Cone} E_\Delta, E_\Delta)^L \subset \prod_{\Delta \in S} \operatorname{Cone} E_\Delta$

$\widetilde{Z}_L =$ universal cover $=$ RAB



As before, fundamental chamber $E_\Delta = G_\Delta$

$K = ([0,\Delta], \Delta)^L \cong ([0,1], 1)^L$

$\mathcal{C}' = \{\text{chambers in } \prod_\Delta \operatorname{Cone} E_\Delta\} = \prod E_\Delta$

$\mathcal{C} = \{\text{chambers in } \widetilde{Z}_L\}$

$Z_L = D(\mathcal{C}', K) = (\mathcal{C}' \times K)/\sim$

$\widetilde{Z}_L = D(\mathcal{C}, K) = (\mathcal{C} \times K)/\sim \; = |\mathcal{C}|$

Blds for RAAG $=$ Deligne cx for RAAG

Special case: each $E_\Delta = \mathbb{Z} = \langle \Delta \rangle$

Then $\mathcal{C}' = (\mathbb{Z})^S = A'$

$\mathcal{C} = A_L$ the RAAG,

$\widetilde{Z} \ldots = (A_L \times K)/\sim$

$Z_L = D(A_L, K)$ is the bldg of A_L
also = Deligne complex of A_L.

Comments about Buildings

1) A combinatorial "building" is a set $\mathcal{C}$ (of "chambers") and a Coxeter system (W, S) + ...

$\delta : \mathcal{C} \times \mathcal{C} \to W$

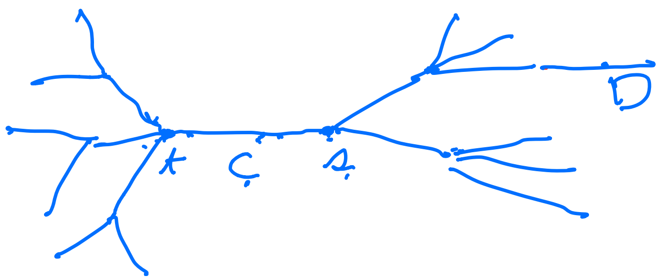
2) For each $s \in S$, there is equivalence relation on $\mathcal{C}$. Two chambers $C, D \in \mathcal{C}$ are <u>s-adjcent</u> if they are in same equivalence class (and $C \neq D$). Put

$$q_s = q_s(C) = \# \{ D \mid D \text{ s-adj to } C \}$$

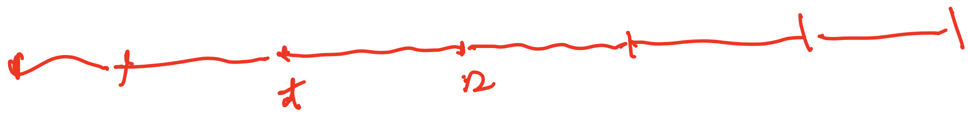
(This is ind of C if $\mathcal{C}$ has chamber transitive automorphism gp.)
Bldg $\mathcal{C}$ is <u>thick</u> if each $q_s \geq 2$.

3. Basic example: $\mathcal{C} = \{$edges in a tree$\}$

$(W,S) = (D_\infty, \{s,t\})$



$\underline{s}$ = word s, t
$w(\underline{s}) \in W$



Standard examples of locally finite bldgs

1) Spherical bldgs : (W,S) irreducible spherical gp of rank ≥ 3, thick

Tits' Thm $\mathcal{C}$ as above (thick, spherical bldg) then $\mathcal{C}$ comes from an algebraic gp over finite field $\mathbb{F}_q$, e.g. $PGL_n(\mathbb{F}_q)$ = chains of subspaces in $(\mathbb{F}_q)^n$

Continue to new note

2) Generalized m-gons = rank 2 spherical blds

Def Bipartite graph of girth $2m$ and diameter m.

$(W, S) = (D_m, \{s, t\})$

Feit-Higman: Finite graph $\Rightarrow$ $m \in \{2, 3, 4, 6, 8\}$

$\mathbb{F}_q = \mathbb{F}_2$

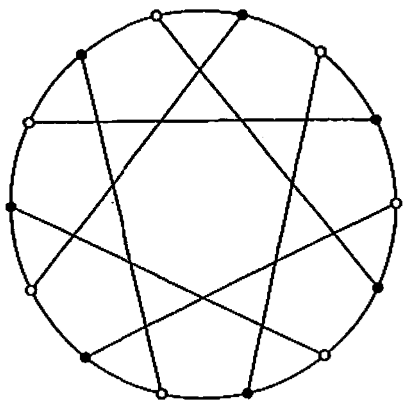


Fig. 4.1. The incidence graph of the Fano plane.

spherical bldg

3. <u>Affine Bldgs</u> (W, S) is a Euclidean Coxeter gp. (irreducible $\Rightarrow K =$ simplex). Use Fields with valuations, e.g., $SL_n(\mathbb{F}_q((t)))$

4. Kac-Moody groups and buildings over $\mathbb{F}_q$. Many possibilities for Coxeter system (W,S) can occur, but not all. Need "generalized Cartan matrix" with integer entries. This $\Rightarrow$

$$4\cos^2 \frac{\pi}{m_{st}} \in \mathbb{Z}$$

$$\Rightarrow \quad m_{st} \in \{2, 3, 4, 6, \infty\}$$

Main Construction
(Generalizing Polyhedral Product)

<u>Simplified Assumption:</u> $(W', S) =$ spherical Cox system

Nerve $(W', S) = \Delta =$ a simplex

$L < \Delta$ is flag subcomplex same vertex set

$f: L \longrightarrow \Delta$ inclusion. New Coxeter Matrix

We have omitted edges from L.

$$m_{st} = \begin{cases} m_{f(s)f(t)} & ; \text{if } \{s,t\} \in L'^{(1)} \\ \infty & \text{otherwise} \end{cases}$$

(W, S) = resulting Coxeter system

K' = Cone Δ = chamber for (W', S)

K = Cone $L(W,S)$ = chamber for (W,S)

$\hat{f} : W \to W'$ homo defined by $s \to f(s)$.

(Actual Assumption: $L' = L(W', S')$

$f : L \to L'$ simpl map

injective on each simplex

(m_{st}) defined by above formula. This defines (W,S)

then L' is a metric flag cx

So each edge of L has length

Assumption: L is a metric flag cx.

$$D(W', K') = \text{disk} = \text{Coxeter cx for } W'$$

$(W' \times \cancel{K})/\sim$ Cone on

like cube

$$X_L = D(W', K) = (W' \times K)/\sim \quad , K = \text{fund chamber for } (W,S)$$

Note: not simply connected

(Because we have omitted 2-cells corresponding to deleted edges)

$\tilde{D}$ = univ cover of $D(W', K)$

W = gp of all lifts of W'-action to $\widetilde{D}$ and

$\widetilde{D} = \Sigma(W, S)$. $= D(\widetilde{W}, K)$

Artin gps: Same argument works

A' = spherical Artin gp

$\Lambda' = D(A', K')$ = Deligne cx for A'

$=$ gp of lifts

A = Artin gp of type FC (because L = flag cx)

$D(A', K)$ is not simply connected but NPC.

$\overline{D(A', K)} = D(A, K)$ = Deligne cx for A.

"?"

is this contractible?

Buildings: $\mathcal{C}'$ = spherical bldg of type (W', S')

$|\mathcal{C}'| := D(\mathcal{C}', K')$

Can also form $D(\mathcal{C}', K)$
not simply connected.

Thm Let $\mathcal{C} = \left\{ \begin{array}{l} \text{inverse images of centers of chambers} \\ \text{in } \widetilde{D(\mathcal{C}', K)} \end{array} \right\}$

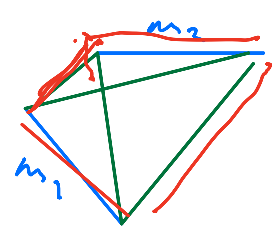
Then $\mathcal{C}$ is bldg of type (W, S):

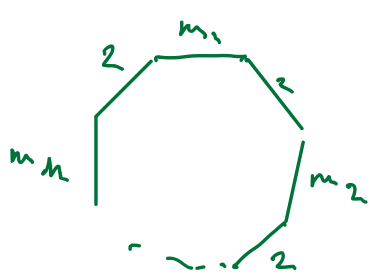
$$|\mathcal{C}| = \widetilde{D(\mathcal{C}', K)} = D(\mathcal{C}, K).$$

Can the $\mathbb{F}_q$ be different

Example: $W = D_{m_1} \times D_{m_2} \times \dots \times D_{m_k}$

$L =$ 2k gon





We can put generalized m_i-gon over each labelled

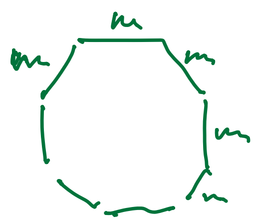
and the $s_i \overset{m_i}{\text{———}} t_i$ and the

$\mathbb{F}_q$ over different edges can be different

mixed type

Example $L =$ $2k$-gon

fold L onto a single edge labeled m. Get bldg of type



Lattices in Aut $(\mathcal{C})$

Suppose $\mathrm{Aut}\,\mathcal{C} \curvearrowright \mathcal{C}$ is transitive. If $\mathcal{C}$ is thick and not spherical, then $\mathrm{Aut}(\mathcal{C})$ will be locally compact, not discrete.

$\mathcal{C}' =$ finite spherical building.

Then $|\mathrm{Aut}(\mathcal{C}')| < \infty$, so trivial subgp will be uniform lattice in $G' = \mathrm{Aut}(\mathcal{C}')$

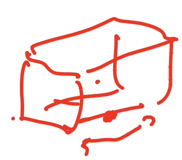
$\mathcal{D}(\mathcal{C}, K)$

The group G of all lifts of G' to $\mathrm{Aut}(\mathcal{C})$ will be chamber transitive

The gp Γ of all lifts of $\{1\}$ to $\mathrm{Aut}(\mathcal{C})$ is $\Gamma = \pi_1(D(\mathcal{C}; K))$, if is a torsion-free lattice in G.

Good to produce group-actions on CAT(0) - cube complexes

$\widetilde{X}_L$

X_L  $\circlearrowleft (C_2)^{?}$