Mod p and torsion homology growth in nonpositive curvature

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Lück Approximation Theorem

Let X be a finite CW-complex. Let $\Gamma = \pi_1(X)$. We will always assume that Γ is *residually finite*.This means there is a collection of subgroups

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots$$

so that each Γ_i is finite index and normal in Γ, and $\cap \Gamma_i = 1$.

Let X_k denote the finite regular cover of X corresponding to Γ_k.

Theorem (Lück)

$$\lim_{k \to \infty} \frac{b_i(X_k, \mathbb{Q})}{[\Gamma : \Gamma_k]} = b_i^{(2)}(\widetilde{X}, \Gamma)$$

The $b_i^{(2)}(\widetilde{X}, \Gamma)$ are the L^2-Betti numbers of $\widetilde{X}$ with respect to Γ-action.

mod p L^2-Betti numbers

From now on, my spaces will be aspherical. I will leave off X from the notation and talk about groups: $b_k(\Gamma) = b_k(B\Gamma)$.

Given a residually finite group Γ, we fix a sequence Γ_i as before and define the $\mathbb{F}_p - L^2$*-Betti number* of Γ to be:

Definition

$$b_i^{(2)}(\Gamma; \mathbb{F}_p) := \limsup_k \frac{b_i(\Gamma_k; \mathbb{F}_p)}{[\Gamma : \Gamma_k]}$$

We don't know if this depends on the chain, but I am leaving it off of the notation. By the universal coefficient theorem, we have that

$$b_i^{(2)}(\Gamma; \mathbb{F}_p) \geq b_i^{(2)}(\Gamma; \mathbb{Q})$$

Sample computations

Free groups and fundamental groups of closed surfaces have $b_i^{(2)}(\Gamma, \mathbb{F}_p)$ concentrated in dimension 1.

Sometimes, there are models for the $B\Gamma_i$ where the number of cells grows sublinearly in the index. In this case, all mod p L^2-Betti numbers vanish. For example, this happens for $\mathbb{Z}^n$.

Conjecture (Lück)

$$b_i^{(2)}(\Gamma; \mathbb{F}_p) = b_i^{(2)}(\Gamma; \mathbb{Q})$$

Some conjectures

All computations of L^2-Betti numbers agree with the following conjecture attributed to Singer.

Singer Conjecture

Let M^n be a closed aspherical manifold. Then

$$b_i^{(2)}(\pi_1(M^n);\mathbb{Q}) = 0 \text{ for } i \neq \frac{n}{2}.$$

Lück Conjecture

Let M^n be an odd-dimensional closed aspherical manifold. Let $\Gamma_k \lhd \pi_1(M^n)$ be any normal chain with $\bigcap_k \Gamma_k = 1$. If $i \neq (n-1)/2$ then

$$\lim_{k\to\infty} \frac{\log |H_i(\Gamma_k;\mathbb{Z})_{tors}|}{[\pi_1(M^n):\Gamma_k]} = 0.$$

Some conjectures

Earlier, Bergeron and Venkatesh had made a similar conjecture for vanishing of homological torsion growth for locally symmetric spaces and specific chains Γ_k. So Lück is conjecturing that this phenomenon holds for all closed aspherical manifolds and all chains.

In both cases, the conjectures are more precise and that if $i = (n-1)/2$ predict non-vanishing of torsion growth in certain cases

More conjectures

The last two conjectures imply an $\mathbb{F}_p$-version of the Singer Conjecture.

$\mathbb{F}_p$-Singer Conjecture

Let M^n be a closed aspherical n-manifold with residually finite fundamental group. Then

$$b_i^{(2)}(\pi_1(M^n); \mathbb{F}_p) = 0 \text{ for } i \neq \frac{n}{2}.$$

The implication follows from Künneth formula and universal coefficients. Suppose there exists M^n with

$$b_i^{(2)}(\pi_1(M^n); \mathbb{F}_p) > b_i^{(2)}(\pi_1(M^n); \mathbb{Q}) = 0 \text{ for } i > \frac{n}{2}$$

Then there would be exponential torsion growth in dimension i or $i-1$. If this happens to be the predicted dimension, then Künneth formula will push nontrivial mod p Betti numbers away from middle dimension.

Main Theorem

Theorem

For any odd prime p, the $\mathbb{F}_p$-Singer Conjecture fails in all odd dimensions ≥ 7 and all even dimensions ≥ 14.

Our manifolds are constructed using right-angled Coxeter groups. They are locally CAT(0), not locally symmetric. Our methods won't work for $p = 2$.

Outline of construction

We first find a group Γ with $b_i^{(2)}(\Gamma; \mathbb{F}_p) > b_i^{(2)}(\Gamma; \mathbb{Q}) = 0$. The group Γ will be a right-angled Artin group.

These groups are far from being fundamental groups of closed aspherical manifolds.

We then realize Γ as a subgroup of a right-angled Coxeter group, which has a finite index subgroup which is π_1 of closed aspherical manifold M^n.

Main point: If $p \neq 2$, we can arrange that $b_i^{(2)}(\Gamma, \mathbb{F}_p) \neq 0$ for $i > \frac{n}{2}$.

Induction and Mayer-Vietoris guarantees that $\mathbb{F}_p$-Singer Conjecture either fails for M^n or for lower (odd)-dimensional example.

Mod p Betti numbers of RAAG's

Here is computation for RAAG's.

Theorem

Let A_L be a right-angled Artin group with defining flag complex L and $k = \mathbb{Q}$ or $\mathbb{F}_p$. Then

$$b_i^{(2)}(A_L; k) = \bar{b}_{i-1}(L; k),$$

where $\bar{b}_{i-1}(L; k)$ is the reduced i^{th} Betti number of L with k-coefficients.

The $\mathbb{Q}$-case was done earlier by Davis and Leary. So if L is a flag triangulation of $\mathbb{R}P^2$, then we have

$$b_3^{(2)}(A_{\mathbb{R}P^2}, \mathbb{Q}) = 0 \text{ and } b_3^{(2)}(A_{\mathbb{R}P^2}, \mathbb{F}_2) = 1$$

This implies exponential torsion growth for $H_2(A_{\mathbb{R}P^2}, \mathbb{Z})$.

Mod p Betti numbers of RAAG's

The computation of $b_i^{(2)}(A_L, \mathbb{F}_p)$ follows from a Mayer-Vietoris spectral sequence (think Acyclic Covering Lemma).
The Salvetti complex BA_L is constructed by gluing together tori according to simplices of L, and we can cover by maximal tori.

In the cover of BA_L corresponding to Γ_k, each torus lifts to a disjoint union of tori, so the only way the Betti numbers can grow linearly is if the number of lifts grows linearly.

Mod p Betti numbers of RAAG's

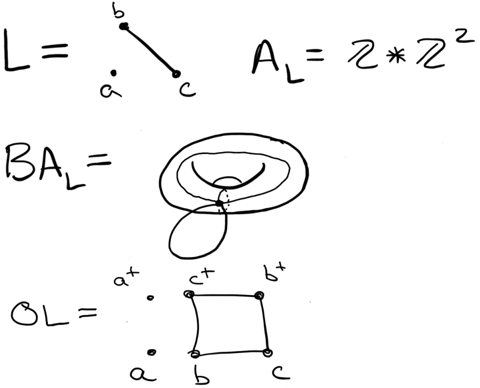
Since the Γ_k's are normal and residual, there are a sublinear number of lifts of each torus (except for the basepoint). In particular, the number of lifts is the ratio of indices

$$\frac{[A_L : \Gamma_k]}{|\pi_1 T_\sigma : \pi_1 T_\sigma \cap \Gamma_k|}$$

In the spectral sequence, the only terms that matter (up to sublinear error) are the collections of maximal tori which intersect in the basepoint, so the only terms that matter are in a row. It turns out this row is exactly homology of L shifted by a degree.

Octahedralizations

The Salvetti complex BA_L has a unique vertex, and the link of that vertex in BA_L is a flag complex denoted OL. It is obtained by "doubling the vertices" of L; take copies L^+ and L and put in simplices iff they project to simplices in L.



Facts about A_L and W_L

If L is a flag complex, then the right-angled Coxeter group W_L has finite index subgroup which is fundamental group of locally CAT(0) cube complex Y_L, where the link of every vertex in Y_L is isomorphic to L.

Theorem

Let L be a flag complex.

1. A_L *is commensurable to* W_{OL}. *(Davis-Januszkiewicz)*
2. *If* T *is a flag triangulation of* S^{n-1}, *then* W_T *has finite index subgroup which is fundamental group of closed aspherical n-manifold.*

Therefore, if OL embeds into a flag triangulation T of S^{n-1}, we get a finite index subgroup of A_L which is subgroup of W_T. We want $b_i^{(2)}(A_L, \mathbb{F}_p) \neq 0$ for $i > \frac{n}{2}$, so it suffices to find a d-complex L with $b_d(L, \mathbb{F}_p) \neq 0$ so that OL embeds into S^{2d}.

Embedding OL

Usually, OL is hard to embed into Euclidean space...for example if $L = S^0 * S^0 * \cdots * S^0$ then $OL = 4pts * 4pts * \cdots * 4pts$.

With Davis, we proved the following theorem.

Theorem (ADOS)

Suppose L is a d-dimensional flag complex, $d \neq 2$.

1. *If $H_d(L, \mathbb{Z}/2) \neq 0$, then OL does not embed into S^{2d}.*
2. *If $d \neq 2$ and $H_d(L, \mathbb{Z}/2) = 0$ then OL embeds as a full subcomplex into a flag PL-triangulation of S^{2d}.*

The theorem comes from computing an obstruction due to van Kampen to embedding d-complexes into S^{2d}. This is a complete obstruction except for 2-complexes embedding into $\mathbb{R}^4$.
Essentially, $p = 2$ comes up because this obstruction is order 2.

Embedding OL

Now, for $p \neq 2$, let L be a flag triangulation of a complex obtained by gluing a 3-disk to a 2-sphere along a degree p map.
Our computation for mod p Betti numbers of RAAG's gives that

$$b_4^{(2)}(A_L; \mathbb{F}_p) = 1.$$

Since $H_3(L; \mathbb{F}_2) = 0$, OL embeds into a flag PL triangulation T of S^6 as a full subcomplex, and the associated Davis complex Σ_T is a contractible 7-manifold.
Induction + Mayer-Vietoris shows that $b_4^{(2)}(W_T, \mathbb{F}_p) \neq 0$ or $\mathbb{F}_p$-Singer conjecture fails in lower dimension.

Fibering

Davis and Okun made following conjecture in analogy with Thurston's virtual fibering conjecture

Conjecture

Suppose that L is a flag triangulation of S^{2n}. Then Y_L has a finite cover which fibers over S^1.

Our manifolds will not virtually fiber over S^1, since such a fibering would imply that many chains have vanishing mod p L^2-Betti numbers for all p.

Question

Suppose that L is a flag triangulation of S^{2n}. Does W_L have a finite index subgroup which surjects $\mathbb{Z}$ and kernel is $FP(\mathbb{Q})$?