# Constructing Bisimplices

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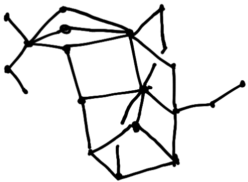
# Motivation

## Notion

Quadric complexes are simply connected square complexes satisfying a certain local combinatorial nonpositive curvature condition. They generalize CAT(0) square complexes and are in many ways analogous to systolic complexes.

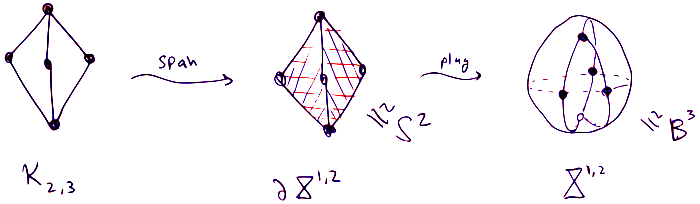


A quadric complex

## Theorem (H.)

*Spanning a "bisimplex" on each $K_{m+1,n+1}$ subgraph, with $m, n \geq 1$, of a locally finite "quadric complex" results in a contractible complex with the same 2-skeleton.*

# Definition?




## Definition

A $(1,1)$-*bisimplex* $⧖^{1,1}$ is a square with its usual cell structure.

An $(m,n)$-*bisimplex* $⧖^{m,n}$, with $m, n \geq 1$, is obtained from $K_{m+1,n+1}$ by first spanning a single $⧖^{m',n'}$ on each proper $K_{m'+1,n'+1}$ subgraph with $m', n' \geq 1$ to obtain $\partial ⧖^{m,n}$ and then plugging *the resulting* $(m+n-1)$-*sphere* with an $(m+n)$-cell.

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In particular, the $(m,n)$-bisimplex $⧖^{m,n}$ is a cellulated ball $B^{m+n}$ such that

- the 1-skeleton is $K_{m+1,n+1}$
- there is exactly one higher cell for each $K_{m'+1,n'+1}$ subgraph with $m',n' \geq 1$ and this cell is a $⧖^{m',n'}$

Compare with the $n$-simplex $\Delta^n$ which is a cellulated $B^n$ such that

- the 1-skeleton is $K_{n+1}$
- there is exactly one simplex for each $K_{n'}$ subgraph

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An $(m,n)$-*bisimplex* $\mathbb{X}^{m,n}$, with $m, n \geq 1$, is obtained from $K_{m+1,n+1}$ by first spanning a single $\mathbb{X}^{m',n'}$ on each proper $K_{m'+1,n'+1}$ subgraph with $m', n' \geq 1$ to obtain $\partial\mathbb{X}^{m,n}$ and then plugging *the resulting* $(m+n-1)$-*sphere* with an $(m+n)$-cell.

## Problem!

The inductive step of the definition crucially depends on $\partial\mathbb{X}^{m,n}$ being homeomorphic to $S^{m+n-1}$.

We will spend the rest of this talk proving that we do indeed have $S^{m+n-1}$ in this inductive step.

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# Morse matchings

## Terminology

In this talk, *cell complex* means regular CW complex. A CW complex is *regular* if its attaching maps are all embeddings.

Not regular

Regular

## Definition

A *Forman discrete Morse matching* on a cell complex $X$ is a collection of pairs of cells $\{\sigma_i \to \tau_i\}_i$ such that:

1. $\sigma_i$ is a codimension 1 face of $\tau_i$; we denote this by $\sigma_i \prec \tau_i$
2. each cell appears in at most one pair
3. there is no cycle of the form: $\sigma_{i_1} \to \tau_{i_1} \succ \sigma_{i_2} \to \tau_{i_2} \succ \cdots \succ \sigma_{i_k} \to \tau_{i_k} \succ \sigma_{i_1}$

A *critical cell* of a Morse matching is one not appearing in any pair.

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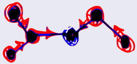
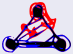
## Nonexamples

## Examples

# The Forman Sphere Theorem

## Examples





## Theorem (Forman)

*If the union of the relative interiors of the critical cells of $X$ forms a subcomplex $Y$ then $X$ collapses onto $Y$.*

## Theorem (Forman)

*For each $d$, let $N_d$ be the number of critical cells of $X$ of dimension $d$. Then $X$ is homotopy equivalent to a CW complex with $N_d$ cells of dimension $d$.*

## Theorem (Forman Sphere Theorem)

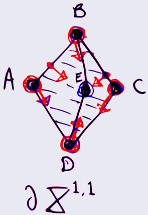
*If $X$ has two critical cells $\sigma_1$ and $\sigma_2$ then $\min_i \dim \sigma_i = 0$ and $X$ is homotopy equivalent to a sphere of dimension $\max_i \dim \sigma_i$.*

# The Forman Sphere Theorem

## Theorem (Forman Sphere Theorem)

*If $X$ has two critical cells $\sigma_1$ and $\sigma_2$ then* $\min_i \dim \sigma_i = 0$ *and $X$ is homotopy equivalent to a sphere of dimension* $\max_i \dim \sigma_i$.

## Example



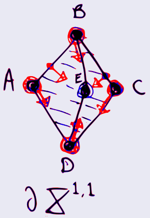

The critical cells are

1. the vertex $E$
2. the square $ABCD$

so $\partial \mathbb{X}^{1,1} \simeq S^2$.

# Proof strategy

## Example




The critical cells are

1. the vertex $E$
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so $\partial\Xi^{1,1} \simeq S^2$.

## Fact

*We can describe a Morse matching with two critical cells for $\partial\Xi^{m,n}$ in general!*

## Problem

This will only tell us that $\partial\Xi^{m,n}$ is homotopy equivalent to a sphere. We need to show that $\partial\Xi^{m,n}$ is homeomorphic to a sphere.

# Proof strategy

**Fact**

*We can describe a Morse matching with two critical cells for $\partial \mathbb{X}^{m,n}$ in general!*

**Problem**

This will only tell us that $\partial \mathbb{X}^{m,n}$ is homotopy equivalent to a sphere. We need to show that $\partial \mathbb{X}^{m,n}$ is homeomorphic to a sphere.

**Theorem (Generalized Poincaré Conjecture)**

*If a topological manifold is homotopy equivalent to a sphere then it is homeomorphic to a sphere.*

How do we prove that a cell complex is a manifold? Something about *links* being *spheres*? Shall we induct on links?

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# The Sphere Recognition Theorem

## Theorem (H.)

*Let $X$ be a cell complex "with links." If $X$ has a Morse matching with two critical cells and the "link" of every cell of $X$ also has such a Morse matching then $X$ is homeomorphic to a sphere.*

## Fact

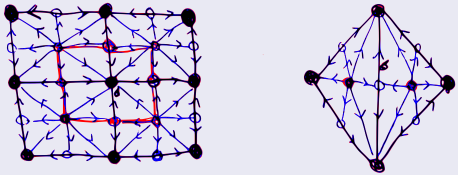
*We can ensure that $\partial \mathbb{X}^{m,n}$ "has links" and can describe such Morse matchings. So this theorem is all we need to complete our construction of bisimplices!*

# BS-links and links

## Definition

Let $X$ be a cell complex and let $\overrightarrow{BS}(X)$ be the barycentric subdivision of $X$ viewed as a "directed simplicial complex." The *BS-link* of a cell $\sigma$ of $X$, denoted $\mathrm{bslink}(\sigma)$, is the full subcomplex of $\overrightarrow{BS}(X)$ induced by the barycenters of all cells of $X$ containing $\sigma$.

## Example



## Definition

If $\mathrm{bslink}(\sigma)$ is $\overrightarrow{BS}(Y)$ for some cell complex $Y$ then $Y$ is the *link* of $\sigma$, denoted $\mathrm{link}(\sigma)$. If $\mathrm{link}(\sigma)$ exists for every cell $\sigma$ of $X$ then $X$ *has links*.

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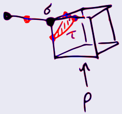
## Remark

If $\sigma$ is a $d$-cell and $\mathrm{link}(\sigma)$ exists then the $(k-d-1)$-cells of $\mathrm{link}(\sigma)$ are in natural bijection with the $k$-cells of $X$ that contain $\sigma$.

# Ensuring that we have links

## Remark

The boundary of a cell $\tau$ of link($\sigma$) is the link of $\sigma$ in the boundary of the cell $\rho$ of $X$ corresponding to $\tau$.




$$\partial\tau = \text{link}_{\partial\rho}(\sigma)$$

So if $X$ has links then so does $\partial\rho$, for every cell $\rho$ of $X$, and the links of the cells of $\partial\rho$ are all homeomorphic to spheres. In fact, this condition is allso sufficient. So if $\partial\rho$ has links for every cell $\rho$ of $X$ then $X$ has links.

This is how we ensure that $\partial\mathbb{X}^{m,n}$ has links and thus apply the Sphere Recognition Theorem!

## Fact

*If $X$ has links then the links of simplices of $\overrightarrow{BS}(X)$ are joins of BS-links and barycentric subdivisions of boundaries of cells of $X$. So if $X$ has spherical links then $X$ is a manifold.*

# Proving the Sphere Recognition Theorem

## Theorem (H.)

*Let $X$ be a cell complex with links. If $X$ has a Morse matching with two critical cells and the link of every cell of $X$ also has such a Morse matching then $X$ is homeomorphic to a sphere.*

## Proof of the Sphere Recognition Theorem.

By the Forman Sphere Theorem, the cell complex $X$ is homotopy equivalent to a sphere. We proceed now by induction. If $\dim X = 0$ then $X$ is a discrete space and so is a 0-dimensional manifold so must be the 0-sphere.

Suppose now that $\dim X = d > 0$. We will prove that $X$ is a manifold. For a given cell $\sigma$ of $X$ we need to show that $\mathrm{link}(\sigma)$ is homeomorphic to a sphere. Let $\tau$ be a cell of $\mathrm{link}(\sigma)$ and let $\rho$ be the corresponding cell of $X$. Then $\mathrm{bslink}(\tau)$, taken in $\mathrm{link}(\sigma)$, is isomorphic to $\mathrm{bslink}(\rho)$. Hence $\mathrm{link}(\sigma)$ has links and has a Morse matching with two critical cells and its links have Morse matchings with two critical cells. Then, by induction, we have that $\mathrm{link}(\sigma)$ is homeomorphic to a sphere. Hence $X$ is a manifold. So, by the Generalized Poincaré Conjecture, $X$ is homeomorphic to a sphere. □

# Fin