Constructing Bisimplices

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Notion

Quadric complexes are simply connected square complexes satisfying a certain local combinatorial nonpositive curvature condition. They generalize CAT(0) square complexes and are in many ways analogous to systolic complexes.

A quadric complex

Theorem (H.)

Spanning a “bisimplex” on each $K_{m+1,n+1}$ subgraph, with $m, n \geq 1$, of a locally finite “quadric complex” results in a contractible complex with the same 2-skeleton.
A $(1, 1)$-bisimplex $\Delta^{1, 1}$ is a square with its usual cell structure.

An $(m, n)$-bisimplex $\Delta^{m,n}$, with $m, n \geq 1$, is obtained from $K_{m+1,n+1}$ by first spanning a single $\Delta^{m',n'}$ on each proper $K_{m'+1,n'+1}$ subgraph with $m', n' \geq 1$ to obtain $\partial \Delta^{m,n}$ and then plugging the resulting $(m + n - 1)$-sphere with an $(m + n)$-cell.
Definition

A \((1, 1)\)-bisimplex \(\Delta^{1,1}\) is a square with its usual cell structure.

An \((m, n)\)-bisimplex \(\Delta^{m,n}\), with \(m, n \geq 1\), is obtained from \(K_{m+1,n+1}\) by first spanning a single \(\Delta^{m',n'}\) on each proper \(K_{m'+1,n'+1}\) subgraph with \(m', n' \geq 1\) to obtain \(\partial \Delta^{m,n}\) and then plugging the resulting \((m + n - 1)\)-sphere with an \((m + n)\)-cell.

In particular, the \((m, n)\)-bisimplex \(\Delta^{m,n}\) is a cellulated ball \(B^{m+n}\) such that

- the 1-skeleton is \(K_{m+1,n+1}\)
- there is exactly one higher cell for each \(K_{m'+1,n'+1}\) subgraph with \(m', n' \geq 1\) and this cell is a \(\Delta^{m',n'}\)

Compare with the \(n\)-simplex \(\Delta^n\) which is a cellulated \(B^n\) such that

- the 1-skeleton is \(K_{n+1}\)
- there is exactly one simplex for each \(K_{n'}\) subgraph
Definition

A (1, 1)-bisimplex $\Box^{1,1}$ is a square with its usual cell structure.

An $(m, n)$-bisimplex $\Box^{m,n}$, with $m, n \geq 1$, is obtained from $K_{m+1,n+1}$ by first spanning a single $\Box^{m',n'}$ on each proper $K_{m'+1,n'+1}$ subgraph with $m', n' \geq 1$ to obtain $\partial \Box^{m,n}$ and then plugging the resulting $(m + n - 1)$-sphere with an $(m + n)$-cell.

Problem!

The inductive step of the definition crucially depends on $\partial \Box^{m,n}$ being homeomorphic to $S^{m+n-1}$.

We will spend the rest of this talk proving that we do indeed have $S^{m+n-1}$ in this inductive step.
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Morse matchings

Terminology
In this talk, cell complex means regular CW complex. A CW complex is regular if its attaching maps are all embeddings.

Not regular

Regular

Definition
A Forman discrete Morse matching on a cell complex $X$ is a collection of pairs of cells $\{\sigma_i \to \tau_i\}_i$ such that:

1. $\sigma_i$ is a codimension 1 face of $\tau_i$; we denote this by $\sigma_i \prec \tau_i$
2. each cell appears in at most one pair
3. there is no cycle of the form: $\sigma_{i_1} \to \tau_{i_1} \succ \sigma_{i_2} \to \tau_{i_2} \succ \cdots \succ \sigma_{i_k} \to \tau_{i_k} \succ \sigma_{i_1}$

A critical cell of a Morse matching is one not appearing in any pair.
Morse matchings

Definition

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A critical cell of a Morse matching is one not appearing in any pair.

Nonexamples

Examples

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The Forman Sphere Theorem

Examples

Theorem (Forman)

If the union of the relative interiors of the critical cells of $X$ forms a subcomplex $Y$ then $X$ collapses onto $Y$.

Theorem (Forman)

For each $d$, let $N_d$ be the number of critical cells of $X$ of dimension $d$. Then $X$ is homotopy equivalent to a CW complex with $N_d$ cells of dimension $d$.

Theorem (Forman Sphere Theorem)

If $X$ has two critical cells $\sigma_1$ and $\sigma_2$ then $\min_i \dim \sigma_i = 0$ and $X$ is homotopy equivalent to a sphere of dimension $\max_i \dim \sigma_i$. 
Theorem (Forman Sphere Theorem)

If $X$ has two critical cells $\sigma_1$ and $\sigma_2$ then $\min_i \dim \sigma_i = 0$ and $X$ is homotopy equivalent to a sphere of dimension $\max_i \dim \sigma_i$.

Example

The critical cells are
1. the vertex $E$
2. the square $ABCD$

so $\partial \Delta^{1,1} \cong S^2$. 
Proof strategy

Example

The critical cells are
1. the vertex $E$
2. the square $ABCD$

so $\partial \boxtimes_{1,1} \cong S^2$.

Fact

We can describe a Morse matching with two critical cells for $\partial \boxtimes^{m,n}$ in general!

Problem

This will only tell us that $\partial \boxtimes^{m,n}$ is homotopy equivalent to a sphere. We need to show that $\partial \boxtimes^{m,n}$ is homeomorphic to a sphere.
Proof strategy

Fact

We can describe a Morse matching with two critical cells for $\partial \Sigma^{m,n}$ in general!

Problem

This will only tell us that $\partial \Sigma^{m,n}$ is homotopy equivalent to a sphere. We need to show that $\partial \Sigma^{m,n}$ is homeomorphic to a sphere.

Theorem (Generalized Poincaré Conjecture)

If a topological manifold is homotopy equivalent to a sphere then it is homeomorphic to a sphere.

How do we prove that a cell complex is a manifold? Something about links being spheres? Shall we induct on links?
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The Sphere Recognition Theorem

**Theorem (H.)**

Let $X$ be a cell complex “with links.” If $X$ has a Morse matching with two critical cells and the “link” of every cell of $X$ also has such a Morse matching then $X$ is homeomorphic to a sphere.

**Fact**

We can ensure that $\partial \bar{\Sigma}^{m,n}$ “has links” and can describe such Morse matchings. So this theorem is all we need to complete our construction of bisimplices!
BS-links and links

Definition

Let $X$ be a cell complex and let $\overrightarrow{BS}(X)$ be the barycentric subdivision of $X$ viewed as a “directed simplicial complex.” The BS-link of a cell $\sigma$ of $X$, denoted $\text{bslink}(\sigma)$, is the full subcomplex of $\overrightarrow{BS}(X)$ induced by the barycenters of all cells of $X$ containing $\sigma$.

Example

![Example Diagram]

Definition

If $\text{bslink}(\sigma)$ is $\overrightarrow{BS}(Y)$ for some cell complex $Y$ then $Y$ is the link of $\sigma$, denoted $\text{link}(\sigma)$. If $\text{link}(\sigma)$ exists for every cell $\sigma$ of $X$ then $X$ has links.
BS-links and links

**Definition**

Let $X$ be a cell complex and let $\overrightarrow{BS}(X)$ be the barycentric subdivision of $X$ viewed as a “directed simplicial complex.” The **BS-link** of a cell $\sigma$ of $X$, denoted $bslink(\sigma)$, is the full subcomplex of $\overrightarrow{BS}(X)$ induced by the barycenters of all cells of $X$ containing $\sigma$.

**Definition**

If $bslink(\sigma)$ is $\overrightarrow{BS}(Y)$ for some cell complex $Y$ then $Y$ is the **link** of $\sigma$, denoted $link(\sigma)$. If $link(\sigma)$ exists for every cell $\sigma$ of $X$ then $X$ has links.

**Remark**

If $\sigma$ is a $d$-cell and $link(\sigma)$ exists then the $(k - d - 1)$-cells of $link(\sigma)$ are in natural bijection with the $k$-cells of $X$ that contain $\sigma$. 
Ensuring that we have links

Remark

The boundary of a cell \( \tau \) of \( \text{link}(\sigma) \) is the link of \( \sigma \) in the boundary of the cell \( \rho \) of \( X \) corresponding to \( \tau \).

\[
\partial \tau = \text{link}_{\partial \rho}(\sigma)
\]

So if \( X \) has links then so does \( \partial \rho \), for every cell \( \rho \) of \( X \), and the links of the cells of \( \partial \rho \) are all homeomorphic to spheres. In fact, this condition is also sufficient. So if \( \partial \rho \) has links for every cell \( \rho \) of \( X \) then \( X \) has links.

This is how we ensure that \( \partial \mathbb{X}^{m,n} \) has links and thus apply the Sphere Recognition Theorem!

Fact

If \( X \) has links then the links of simplices of \( \overrightarrow{BS}(X) \) are joins of BS-links and barycentric subdivisions of boundaries of cells of \( X \). So if \( X \) has spherical links then \( X \) is a manifold.
Theorem (H.)

Let $X$ be a cell complex with links. If $X$ has a Morse matching with two critical cells and the link of every cell of $X$ also has such a Morse matching then $X$ is homeomorphic to a sphere.

Proof of the Sphere Recognition Theorem.

By the Forman Sphere Theorem, the cell complex $X$ is homotopy equivalent to a sphere. We proceed now by induction. If $\dim X = 0$ then $X$ is a discrete space and so is a 0-dimensional manifold so must be the 0-sphere.

Suppose now that $\dim X = d > 0$. We will prove that $X$ is a manifold. For a given cell $\sigma$ of $X$ we need to show that $\text{link}(\sigma)$ is homeomorphic to a sphere. Let $\tau$ be a cell of $\text{link}(\sigma)$ and let $\rho$ be the corresponding cell of $X$. Then $\text{bslink}(\tau)$, taken in $\text{link}(\sigma)$, is isomorphic to $\text{bslink}(\rho)$. Hence $\text{link}(\sigma)$ has links and has a Morse matching with two critical cells and its links have Morse matchings with two critical cells. Then, by induction, we have that $\text{link}(\sigma)$ is homeomorphic to a sphere. Hence $X$ is a manifold.

So, by the Generalized Poincaré Conjecture, $X$ is homeomorphic to a sphere. $\square$
Motivation and Basic Construction
Discrete Morse Theory and Forman’s Sphere Theorem
PL-spheres and Links

Fin

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