Hierarchical hyperbolicity of extra-large Artin groups

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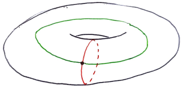
Complexes associated to RAAGs

Simplest (non-abelian) RAAG: Free group on two generators.

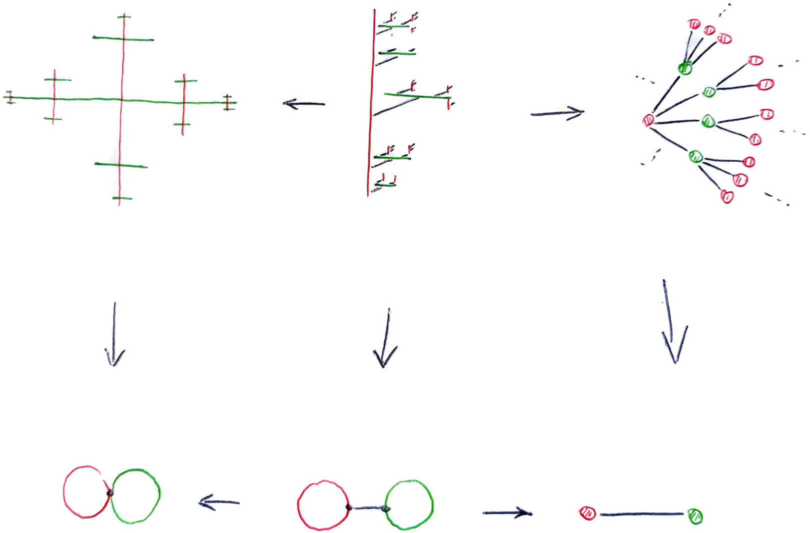
RAAG associated to the graph:

Free product of two copies of $\mathbb{Z} = \pi_1$ of the edge of groups below:

classifying space (*Salvetti complex*) = polyhedral product of circles:



classifying space = complex of spaces over that edge:

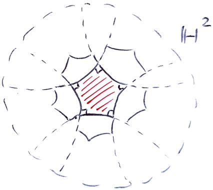


Complexes associated to RAAGs

Generalisation to arbitrary RAAG (or graph product):

Graph Γ $\rightsquigarrow$ "cubical realisation" of the poset of cliques of Γ.

For that Γ, we get the Davis building D_Γ.
(apartment = hyperbolic plane tiled by right-angled pentagons.)



Action on the Davis building

We can use the geometry of the Davis building to study RAAGs and graph products.

(Genevois-M., 2018): Study the automorphism group of graph products over *atomic* graphs:

- computation of the automorphism group,
- geometric features (lack of property (T), acylindrical hyperbolicity)

Key ingredient: Extend the action $G \curvearrowright D_\Gamma$ to an action $\mathrm{Aut}(G) \curvearrowright D_\Gamma$.

From right-angled to general Artin groups

Coxeter diagram: Finite simplicial graph Γ + for every edge ab, an integer $m_{ab} \geq 2$.

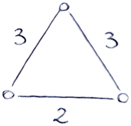
$\hookrightarrow$ **Coxeter group** W_Γ:

$$W_\Gamma = \langle V(\Gamma) \mid a^2 = 1, \quad \underbrace{aba\cdots}_{m_{ab}} = \underbrace{bab\cdots}_{m_{ab}} \rangle.$$

$\hookrightarrow$ **Artin group** A_Γ:

$$A_\Gamma = \langle V(\Gamma) \mid \underbrace{aba\cdots}_{m_{ab}} = \underbrace{bab\cdots}_{m_{ab}} \rangle.$$

$\Gamma =$



- W_Γ = symmetric group S_4
- A_Γ = braid group B_4.

Mysterious Artin groups

Structure:

- Are they torsion-free?
- What is their centre?
- Do they have a soluble word problem?

Geometry:

- Are they non-positively curved?
- Do they have hyperbolic features?

Open in general, partial answers for nice families (finite type, dim. 2, type FC, etc.)

Hyperbolic features of extra-large Artin groups

Extra-large Artin groups: All $m_{ab} \geq 4$.

What was known, for at least 3 generators:

- Huang-Osajda (2017): They are systolic.
- Haettel (2019): If extra-extra-large (all $m_{ab} \geq 5$), they are CAT(0) and acylindrically hyperbolic.
- M.-Przytycki (2019): They are acylindrically hyperbolic (and have an acylindrical action on a nice hyperbolic complex).

Theorem (Hagen-M.-Sisto, soon)

Extra-large Artin groups are (virtually) hierarchically hyperbolic.

$\hookrightarrow$ They have finite asymptotic dimension.

Looking for a hyperbolic complex

The (modified) Deligne complex: Analogue for Artin groups of the Davis building for graph products, and of the Davis complex for general Coxeter groups.

- vertices: left cosets of subgroups generated by $0, 1$, or 2 generators (with $m_{ab} < \infty$) (in general: left cosets of parabolic subgroups *of finite type*)
- edges: inclusion.
- Deligne complex D_Γ: flag completion.

A_Γ acts cocompactly on D_Γ (but not properly!)

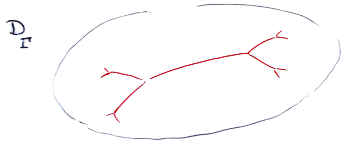
Open in general: Is D_Γ contractible? CAT(0)?
$\hookrightarrow$ would imply the $K(\pi, 1)$ conjecture for Artin groups.

For extra-large Artin groups: D_Γ admits a CAT(-1) metric. (Charney-Davis)

Looking for an acylindrical action...

Acylindrical action: "Few group elements coarsely stabilise far-away points."

Bad news: Standard generators of A_Γ stabilise pointwise unbounded trees!



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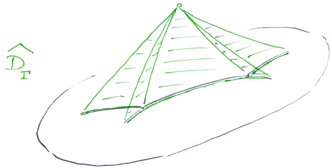
Bad news: Standard generators of A_Γ stabilise pointwise unbounded trees!

$\hookrightarrow$ Naive remedy: Get rid of all these trees. (Cone-off)



Theorem (M.-Przytycki, 2019)

- The cone-off $\widehat{D}_\Gamma$ is CAT(-1).
- The action $A_\Gamma \curvearrowright \widehat{D}_\Gamma$ is acylindrical.

$\hookrightarrow$ Tits Alternative: Subgroups either contain F_2 or are virtually $\{1\}, \mathbb{Z}, \mathbb{Z}^2$.

Beyond acylindrical hyperbolicity

Acylindrical hyperbolicity, relative hyperbolicity, etc.: Hyperbolicity in *some* directions.

Finer notion: Hierarchical hyperbolicity (Behrstock-Hagen-Sisto, 2014)

- hyperbolicity away from some product regions,
- factors of products have also a similar structure (and the process terminates),
- control over how products intersect.

Unifies the geometry of $\mathrm{Mod}(S_g)$ and of RAAGs (and other cubulable groups).

Nice consequences:

- Tits alternative,
- Dehn function at most quadratic,
- finite asymptotic dimension,
- etc.

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2. Background

2.1. **Hierarchically hyperbolic spaces.** In this section we recall the basic definitions and properties of hierarchically hyperbolic spaces as introduced in [BHS17b, BHS15].

Definition 2.1 (Hierarchically hyperbolic space). A q–quasigeodesic space $(\mathcal{X}, \mathsf{d}_{\mathcal{X}})$ is said to be *hierarchically hyperbolic* if there exists $\delta \geqslant 0$, an index set $\mathfrak{S}$, and a set $\{\mathcal{C}W \mid W \in \mathfrak{S}\}$ of δ–hyperbolic spaces $(\mathcal{C}U, \mathsf{d}_U)$, such that the following conditions are satisfied:

(1) **(Projections.)** There is a set $\{\pi_W : \mathcal{X} \to 2^{\mathcal{C}W} \mid W \in \mathfrak{S}\}$ of *projections* sending points in $\mathcal{X}$ to sets of diameter bounded by some $\xi \geqslant 0$ in the various $\mathcal{C}W \in \mathfrak{S}$. Moreover, there exists K so that each π_W is (K,K)–coarsely Lipschitz and $\pi_W(\mathcal{X})$ is K–quasiconvex in $\mathcal{C}W$.

(2) **(Nesting.)** $\mathfrak{S}$ is equipped with a partial order $\sqsubseteq$, and either $\mathfrak{S} = \varnothing$ or $\mathfrak{S}$ contains a unique $\sqsubseteq$–maximal element; when $V \sqsubseteq W$, we say V is *nested* in W. We require that $W \sqsubseteq W$ for all $W \in \mathfrak{S}$. For each $W \in \mathfrak{S}$, we denote by $\mathfrak{S}_W$ the set of $V \in \mathfrak{S}$ such that $V \sqsubseteq W$. Moreover, for all $V, W \in \mathfrak{S}$ with $V \sqsubsetneq W$ there is a specified subset $\rho^V_W \subset \mathcal{C}W$ with $\mathrm{diam}_{\mathcal{C}W}(\rho^V_W) \leqslant \xi$. There is also a *projection* $\rho^W_V : \mathcal{C}W \to 2^{\mathcal{C}V}$.

(3) **(Orthogonality.)** $\mathfrak{S}$ has a symmetric and anti-reflexive relation called *orthogonality*: we write $V \perp W$ when V, W are orthogonal. Also, whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$. Finally, we require that for each $T \in \mathfrak{S}$ and each $U \in \mathfrak{S}_T$ for which $\{V \in \mathfrak{S}_T \mid V \perp U\} \neq \varnothing$, there exists $W \in \mathfrak{S}_T - \{T\}$, so that whenever $V \perp U$ and $V \sqsubseteq T$, we have $V \sqsubseteq W$; we say W is a *container* associated with $T \in \mathfrak{S}$ and $U \in \mathfrak{S}_T$. Finally, if $V \perp W$, then V, W are not $\sqsubseteq$–comparable.

(4) **(Transversality and consistency.)** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested in the other, then we say V, W are *transverse*, denoted $V \pitchfork W$. There exists $\kappa_0 \geqslant 0$ such that if $V \pitchfork W$, then there are sets $\rho^V_W \subseteq \mathcal{C}W$ and $\rho^W_V \subseteq \mathcal{C}V$ each of diameter at most ξ and satisfying:
$$\min\left\{\mathsf{d}_W(\pi_W(x), \rho^V_W), \mathsf{d}_V(\pi_V(x), \rho^W_V)\right\} \leqslant \kappa_0$$
for all $x \in \mathcal{X}$.
For $V, W \in \mathfrak{S}$ satisfying $V \sqsubseteq W$ and for all $x \in \mathcal{X}$, we have:
$$\min\left\{\mathsf{d}_W(\pi_W(x), \rho^V_W), \mathrm{diam}_{\mathcal{C}V}(\pi_V(x) \cup \rho^W_V(\pi_W(x)))\right\} \leqslant \kappa_0.$$
Finally, if $U \sqsubseteq V$, then $\mathsf{d}_W(\rho^U_W, \rho^V_W) \leqslant \kappa_0$ whenever $W \in \mathfrak{S}$ satisfies either $V \sqsubsetneq W$ or $V \pitchfork W$ and $W \not\perp U$.

(5) **(Finite complexity.)** There exists $n \geqslant 0$, the *complexity* of $\mathcal{X}$ (with respect to $\mathfrak{S}$), so that any set of pairwise–$\sqsubseteq$–comparable elements has cardinality at most n.

(6) **(Large links.)** There exist $\lambda \geqslant 1$ and $E \geqslant \max\{\xi, \kappa_0\}$ such that the following holds. Let $W \in \mathfrak{S}$ and let $x, x' \in \mathcal{X}$. Let $N = \lambda \mathsf{d}_W(\pi_W(x), \pi_W(x')) + \lambda$. Then there exists $\{T_i\}_{i=1,\dots,\lfloor N \rfloor} \subseteq \mathfrak{S}_W - \{W\}$ such that for all $T \in \mathfrak{S}_W - \{W\}$, either $T \in \mathfrak{S}_{T_i}$ for some i, or $\mathsf{d}_T(\pi_T(x), \pi_T(x')) < E$. Also, $\mathsf{d}_W(\pi_W(x), \rho^{T_i}_W) \leqslant N$ for each i.

(7) **(Bounded geodesic image.)** For all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}_W - \{W\}$, and all geodesics γ of $\mathcal{C}W$, either $\mathrm{diam}_{\mathcal{C}V}(\rho^W_V(\gamma)) \leqslant E$ or $\gamma \cap \mathcal{N}_E(\rho^V_W) \neq \varnothing$.

(8) **(Partial Realization.)** There exists a constant α with the following property. Let $\{V_j\}$ be a family of pairwise orthogonal elements of $\mathfrak{S}$, and let $p_j \in \pi_{V_j}(\mathcal{X}) \subseteq \mathcal{C}V_j$. Then there exists $x \in \mathcal{X}$ so that:
- $\mathsf{d}_{V_j}(x, p_j) \leqslant \alpha$ for all j,
- for each j and each $V \in \mathfrak{S}$ with $V_j \sqsubseteq V$, we have $\mathsf{d}_V(x, \rho^{V_j}_V) \leqslant \alpha$, and
- if $W \pitchfork V_j$ for some j, then $\mathsf{d}_W(x, \rho^{V_j}_W) \leqslant \alpha$.

(9) **(Uniqueness.)** For each $\kappa \geqslant 0$, there exists $\theta_u = \theta_u(\kappa)$ such that if $x, y \in \mathcal{X}$ and $\mathsf{d}(x,y) \geqslant \theta_u$, then there exists $V \in \mathfrak{S}$ such that $\mathsf{d}_V(x,y) \geqslant \kappa$.

Notation 2.2. Note that below we will often abuse notation by simply writing $(\mathcal{X}, \mathfrak{S})$ or $\mathfrak{S}$ to refer to the entire package of an hierarchically hyperbolic structure, including all the associated spaces, projections, and relations given by the above definition.

Notation 2.3. When writing distances in $\mathcal{C}U$ for some $U \in \mathfrak{S}$, we often simplify the notation slightly by suppressing the projection map π_U, i.e., given $x, y \in \mathcal{X}$ and $p \in \mathcal{C}U$ we write $\mathsf{d}_U(x,y)$ for $\mathsf{d}_U(\pi_U(x), \pi_U(y))$ and $\mathsf{d}_U(x,p)$ for $\mathsf{d}_U(\pi_U(x), p)$. Note that when we measure distance between a pair of sets (typically both of bounded diameter) we are taking the minimum distance between the two sets. Given $A \subset \mathcal{X}$ and $U \in \mathfrak{S}$ we let $\pi_U(A)$ denote $\cup_{a \in A} \pi_U(a)$.

It is often convenient to work with equivariant hierarchically hyperbolic structures, we now recall the relevant structures for doing so. For details see [BHS15].

Definition 2.4 (Hierarchically hyperbolic groups). Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. The *automorphism group* of $(\mathcal{X}, \mathfrak{S})$ is denoted $\mathrm{Aut}(\mathcal{X}, \mathfrak{S})$ and is defined as follows. An element of $\mathrm{Aut}(\mathcal{X}, \mathfrak{S})$ consists of a map $g : \mathcal{X} \to \mathcal{X}$, together with a bijection $g^\diamondsuit : \mathfrak{S} \to \mathfrak{S}$ and, for each $U \in \mathfrak{S}$, an isometry $g^*(U) : \mathcal{C}U \to \mathcal{C}(g^\diamondsuit(U))$ so that the following diagrams coarsely commute whenever the maps in question are defined (i.e., when U, V are not orthogonal):

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "g"] \arrow[d, "\pi_U"'] & \mathcal{X}' \arrow[d, "\pi_{g^\diamondsuit(U)}"] \\
\mathcal{C}U \arrow[r, "g^*(U)"] & \mathcal{C}(g^\diamondsuit(U))
\end{tikzcd}$$

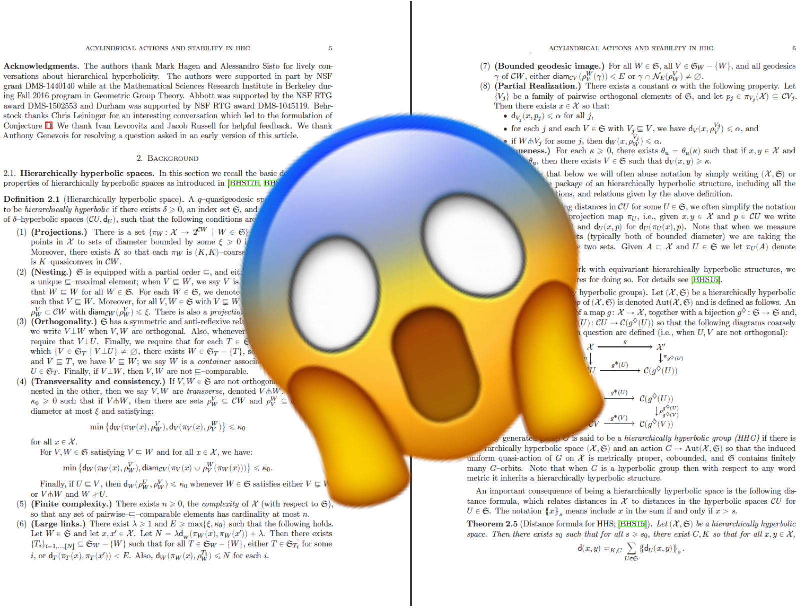
and

$$\begin{tikzcd}
\mathcal{C}U \arrow[r, "g^*(U)"] \arrow[d, "\rho^U_V"'] & \mathcal{C}(g^\diamondsuit(U)) \arrow[d, "\rho^{g^\diamondsuit(U)}_{g^\diamondsuit(V)}"] \\
\mathcal{C}V \arrow[r, "g^*(V)"] & \mathcal{C}(g^\diamondsuit(V))
\end{tikzcd}$$

A finitely generated group G is said to be a *hierarchically hyperbolic group (HHG)* if there is a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ and an action $G \to \mathrm{Aut}(\mathcal{X}, \mathfrak{S})$ so that the induced uniform quasi-action of G on $\mathcal{X}$ is metrically proper, cobounded, and $\mathfrak{S}$ contains finitely many G–orbits. Note that when G is a hyperbolic group then with respect to any word metric it inherits a hierarchically hyperbolic structure.

An important consequence of being a hierarchically hyperbolic space is the following distance formula, which relates distances in $\mathcal{X}$ to distances in the hyperbolic spaces $\mathcal{C}U$ for $U \in \mathfrak{S}$. The notation $\{\!\{x\}\!\}_s$ means include x in the sum if and only if $x > s$.

Theorem 2.5 (Distance formula for HHS; [BHS15]). *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Then there exists s_0 such that for all $s \geqslant s_0$, there exist C, K so that for all $x, y \in \mathcal{X}$,*
$$\mathsf{d}(x,y) \asymp_{K,C} \sum_{U \in \mathfrak{S}} \{\!\{\mathsf{d}_U(x,y)\}\!\}_s.$$



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$$\mathcal{X} \xrightarrow{g} \mathcal{X}' \qquad \downarrow \pi_{g^\diamondsuit(U)}$$

$$\mathcal{C}U \xrightarrow{g^*(U)} \mathcal{C}(g^\diamondsuit(U))$$

$$\xrightarrow{g^*(U)} \mathcal{C}(g^\diamondsuit(U)) \qquad \downarrow \rho^{g^\diamondsuit(U)}_{g^\diamondsuit(V)}$$

$$\mathcal{C}V \xrightarrow{g^*(V)} \mathcal{C}(g^\diamondsuit(V))$$

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An important consequence of being a hierarchically hyperbolic space is the following distance formula, which relates distances in $\mathcal{X}$ to distances in the hyperbolic spaces $\mathcal{C}U$ for $U \in \mathfrak{S}$. The notation $\{\!\{x\}\!\}_s$ means include x in the sum if and only if $x > s$.

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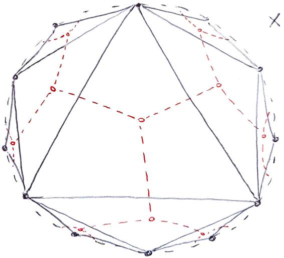
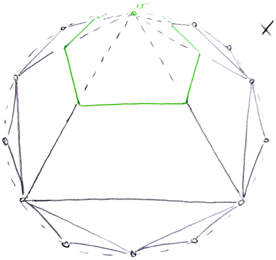
$$\mathsf{d}(x, y) \asymp_{K,C} \sum_{U \in \mathfrak{S}} \{\!\{\mathsf{d}_U(x, y)\}\!\}_s .$$

Action of $Mod(S_{0,4})$ (or $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$) on the Farey complex:



- The complex X is hyperbolic.
- Stabilisers of triangles are trivial. ($\hookrightarrow$ dual graph QI to the Cayley graph.)



- The complex X is hyperbolic.
- Stabilisers of triangles are trivial. ($\hookrightarrow$ dual graph QI to the Cayley graph.)
- For each vertex v, its link is hyperbolic and is a retract of $X - v$ ($\hookrightarrow$ QI embedded in $X - v$).

Encodes the fact that G is hyperbolic relative to the stabilisers of vertices.

Distance formula: "$d_G(x, y) \cong d_X(x, y) + \sum_v d(\pi_{lk(v)}(x), \pi_{lk(v)}(y))$".

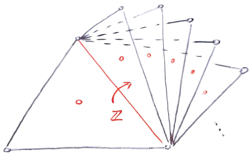
A new approach to hierarchical hyperbolicity

Proto-Theorem: G acting cocompactly on a hyperbolic flag simplicial complex X such that:

- Stabilisers of maximal simplices are finite.
- For each simplex Δ, its link is hyperbolic and QI embedded in $X - \bigcup_{\mathrm{lk}(\Delta')=\mathrm{lk}(\Delta)} \Delta'$.

Then G is hiearchically hyperbolic.

Problem: Links may not be connected.



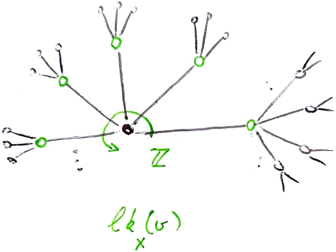
Generalised dual graph (or X-graph): Some simplicial graph W whose vertex set is the set of maximal simplices of X.

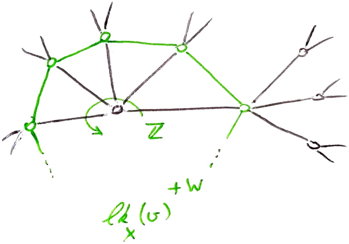
Use W to add new edges:

If Δ, Δ' connected by a W-edge, connect by a W-edge all pairs of vertices of Δ, Δ'.

$\hookrightarrow$ **augmented complex** X^{+W}. (and augmented subcomplexes)

Example: $X =$ Bass-Serre tree of $\mathbb{Z} * \mathbb{Z}$, $W =$ Cayley graph.





Theorem (Behrstock-Hagen-M.-Sisto)

Let G be a group acting cocompactly on a hyperbolic flag simplicial complex X, and inducing a cocompact action on an associated dual graph W, such that:

- Stabilisers of maximal simplices are finite.
- For each simplex Δ, its *augmented* link $\mathrm{lk}_X(\Delta)^{+W}$ is hyperbolic and QI embedded in $\left(X - \bigcup_{\mathrm{lk}(\Delta')=\mathrm{lk}(\Delta)} \Delta'\right)^{+W}$.

$+$ two technical (and removable?) conditions saying that W and X are "well-behaved":

- The poset of links of simplices (for the inclusion) is a "sort of" lattice of finite height, and for every Δ, Δ' their greatest lower bound is of the form $\mathrm{lk}_X(\Delta'')$ with $\Delta \subset \Delta''$.
- For a simplex Δ and two non-adjacent vertices v, v' of $\mathrm{lk}_X(\Delta)$, if v, v' are contained in W-adjacent simplices, they are connected in W-adjacent simplices of $\mathrm{Star}_X(\Delta)$.

Then G is hierarchically hyperbolic. (**Bonus:** The action $G \curvearrowright X$ is acylindrical.)

Question: How to find such an action?

Finding an acylindrical action on a hyperbolic space

Step 1: Find all the maximal subgroups that virtually split as direct products of infinite groups.

Step 2: Study their intersections, and in particular the minimal infinite subgroups obtained.

Step 3: Construct the *commutation graph* of these minimal subgroups:

- vertices $\leftrightarrow$ such minimal infinite intersections.
- edges $\leftrightarrow$ commutation

For hierarchically hyperbolic groups *of an algebraic flavour*, this commutation graph is hyperbolic and the action on it by conjugation is acylindrical (Abbot-Behrstock-Durham).

Example: **RAAG** $\rightsquigarrow$ extension graph of Kim-Koberda.

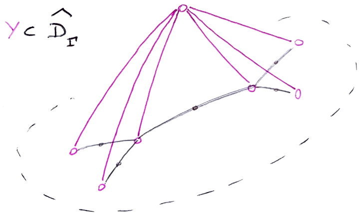
The case of extra-large Artin groups

Maximal subgroups that are virtual products:

- Dihedral parabolic subgroups (virtually $\mathbb{Z} \times F_k$),
- Normalisers of cyclic parabolic subgroups ($= \mathbb{Z} \times F_k$).

Commutation graph obtained: **graph of irreducible parabolic subgroups of finite type.** (Cumplido-Gebhardt-(Gonzáles-Meneses)-Wiest, (Morris-Wright))

isometric to a quasi-dense subgraph Y of the cone-off $\widehat{D}_\Gamma$ (obtained by removing vertices corresponding to cosets of subgroups on ≤ 1 generators)



A ubiquitous graph:

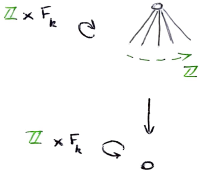
- Generalisation of the curve graph for braid groups.
- Naive construction to make $A_\Gamma \curvearrowright D_\Gamma$ acylindrical.
- Commutation graph of A_Γ.
- At infinity of hyperplane arrangements (Davis-Huang)

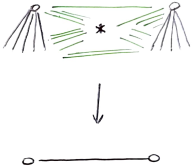
From the commutation graph to the "right" complex

The commutation graph is still not appropriate for our criterion:

- Maximal simplices have infinite stabilisers.
- Some intersections of product subgroups will never appear in links.

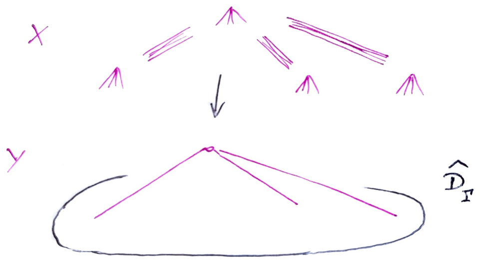
Construction of a **blow-up**:





- Maximal simplices have trivial stabilisers.
- QI to the commutation graph.

This is the complex we apply our criterion to.



Key ingredient. Use the CAT(-1) geometry of $\widehat{D}_\Gamma$ to study the geometry and the relative geometry of links of X.

Your turn to play:

Determine whether your favourite non-positively curved group is hierarchically hyperbolic!

Towards a new definition of hierarchical hyperbolicity?