Geometric model for groups quasi-isometric to RAAGs

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joint with Bruce Kleiner

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QI and ME classification

$f : X_1 \to X_2$ is a quasi-isometry iff there are constants $L, A > 0$ s.t.

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Definition: G and H are *Measure Equivalent* (ME) if there exist commuting, measure-preserving, free actions of G and H on some standard measure space, such that the action of each of the groups F and A admits a finite measure fundamental domain.

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Today: case of right-angled Artin groups

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Isomorphism problem: $G(\Gamma_1) \cong G(\Gamma_2) \Leftrightarrow \Gamma_1 \cong \Gamma_2$ (C. Droms, 1987)

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$X(\Gamma)$ is a union of standard flats.

Questions

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A quasi-action is an "action" by quasi-isometries such that the composition law holds up to bounded distance.

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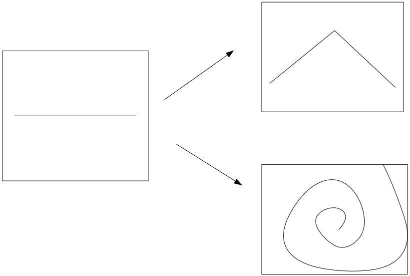

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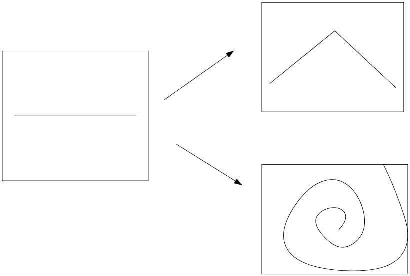

Theorem

Let $\phi : X(\Gamma_1) \to X(\Gamma_2)$ be a quasi-isometry. Then ϕ maps top dimensional flats to top dimensional flats up to finite Hausdorff distance.

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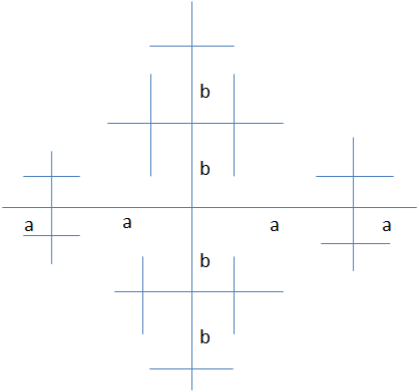

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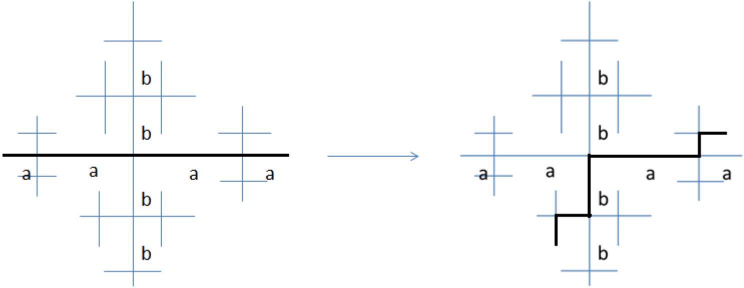

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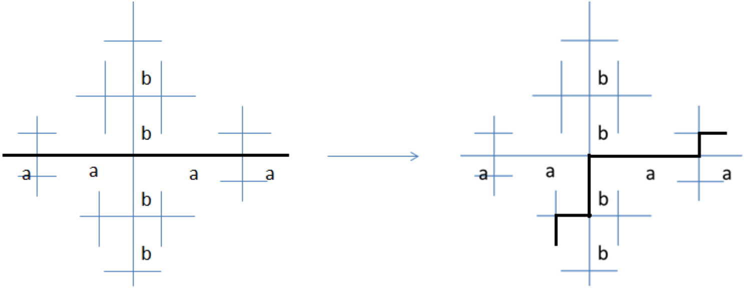

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If there is no transvection in $Aut(G(\Gamma_1))$, then q maps standard flats to standard flats up to finite Hausdorff distance.

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Note that $\bar{q}$ sends vertices in a standard flat bijectively to vertices in a standard flat. Such map is called *flat-preserving* map.

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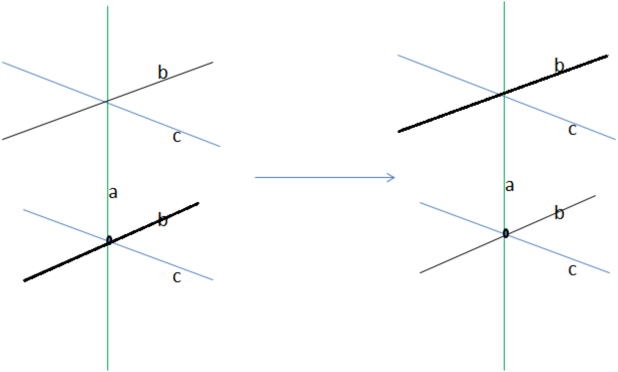

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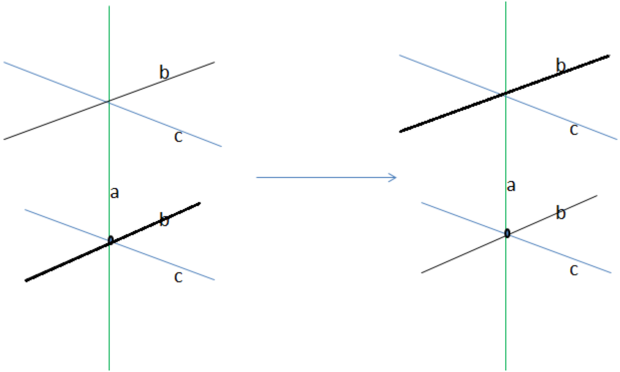

$\{b^i\}_{i\in\mathbb{Z}} \xrightarrow{h} \{ab^i a^{-1}\}_{i\in\mathbb{Z}}$.

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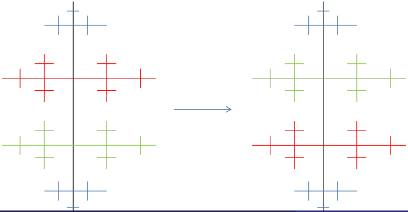
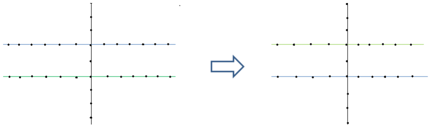
Suppose $Out(G(\Gamma_i))$ is finite for $i = 1, 2$. Then $G(\Gamma_1)$ and $G(\Gamma_2)$ are quasi-isometric if and only if they are isomorphic.

H q.i. to $G(\Gamma) \rightsquigarrow$ quasi-action $H \curvearrowright G(\Gamma)$
We assume $\rho : H \curvearrowright G(\Gamma)$ is an action by flat-preserving bijections.
A flat-preserving map may not be an isometry. It may not respect the order on each standard geodesic line.



Resolve flat-preserving actions



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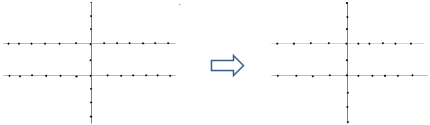

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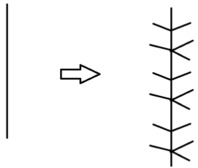
Rough idea: replace each standard line in $X(\Gamma)$ by a "branched line".



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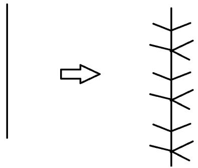

Replace each standard flat in $X(\Gamma)$ by a "branched flat", and glue these branched flats together.

A double fibration

$S(\Gamma)$ – Salvetti complex – Γ-product of circles
$S_e(\Gamma)$ – exploded Salvetti complex – Γ-product of lollipops

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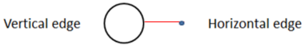

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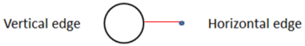

A natural map $S_e(\Gamma) \to S(\Gamma)$ by collapsing all the horizontal edges.

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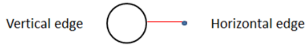

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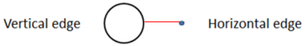

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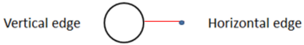

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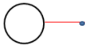

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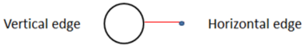

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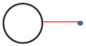

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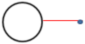

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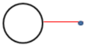

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An example

Take $\Gamma =$ one point.

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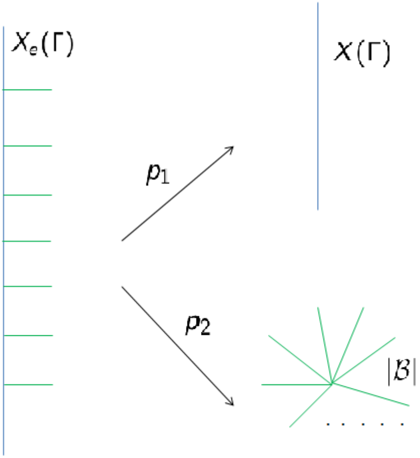

Theorem (H.-Kleiner)

Suppose $Out(G(\Gamma))$ is finite and $G(\Gamma) \neq \mathbb{Z}$. Let $\rho : H \curvearrowright G(\Gamma)$ be a quasi-action. Then ρ is quasi-isometrically conjugate to an isometric action $\rho' : H \curvearrowright X'$ such that

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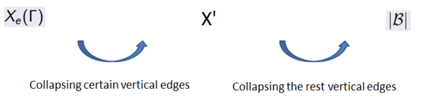

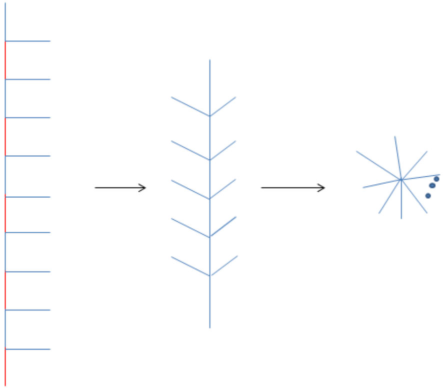

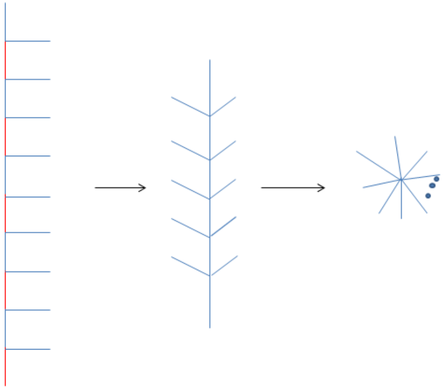

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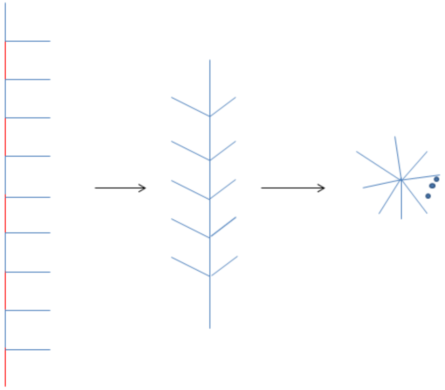

(1) can't collapse all vertical edges, since X' should be q.i. to $G(\Gamma)$ and uniformly locally finite.
(2) When the action does not respect the order along the line, collapsing edges are necessary.

Theorem (H.-Kleiner, 2018)

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Corollary

Suppose $Out(G(\Gamma))$ is finite. If H is quasi-isometric to $G(\Gamma)$, then H acts geometrically on a $CAT(0)$ cube complex.

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Theorem (Horbez-H. 2020)

Suppose $Out(G(\Gamma_i))$ is finite for $i = 1, 2$. Then $G(\Gamma_1)$ and $G(\Gamma_2)$ are measure-equivalence if and only if they are isomorphic.

Thank you!