Some Coupled Supersymmetries and Their Associated Bargmann Transforms

Cameron L. Williams
joint work with Bernhard G. Bodmann and Donald J. Kouri
inspired by John R. Klauder

University of Houston
Departments of Mathematics and Physics

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Given a separable Hilbert space $\mathfrak{H}$ with orthonormal basis $\{e_n\}_{n=0}^{\infty}$, a representation of the oscillator algebra can be found by defining $ae_0 = 0$ and $ae_n = \sqrt{n}e_{n-1}$ for $n > 0$ so that $a^*e_n = \sqrt{n+1}e_{n+1}$ and $a^*ae_n = ne_n$.

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a^* and a are called the creation and annihilation operators, resp.

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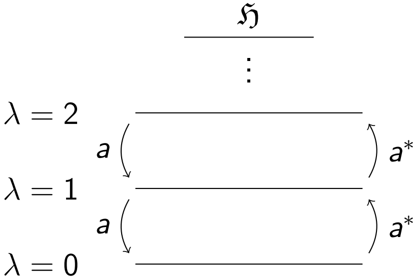
$a^* a$ has eigenvalues $0, 1, 2, \ldots$.

Diagramatically, the action of a and a^* can be realized as



The rungs in this ladder represent different eigenfunctions of a^*a with increasing eigenvalue $(0, 1, 2, \ldots)$.

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Examples include the Bergman space $A^2(\mathbb{D})$ for some bounded domain $\mathbb{D} \subset \mathbb{C}$ and the Segal-Bargmann space. The Segal-Bargmann space is the focus of the next section.

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$$\int_{\mathbb{C}} \frac{d}{dz} f(z) \overline{g(z)}\, d\rho(z, \bar{z}) = \int_{\mathbb{C}} f(z) \overline{z g(z)}\, d\rho(z, \bar{z}).$$

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$$\mathcal{B}f(z) = \frac{1}{\pi^{1/4}} \int_{\mathbb{R}} e^{-z^2/2+\sqrt{2}xz-x^2/2} f(x)\, dx$$

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The Segal-Bargmann space is denoted by $\mathcal{O}L^2(\mathbb{C}, d\rho)$.

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Let $\mathfrak{H}_1, \mathfrak{H}_2$ be Hilbert spaces. Let $a : \mathfrak{H}_1 \to \mathfrak{H}_2$ be closed and densely-defined; similarly, let $b : \mathfrak{H}_2 \to \mathfrak{H}_1$ be closed and densely-defined; and let $\gamma, \delta \in \mathbb{R}$.

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a^*a has eigenvalues $m(\delta - \gamma), m(\delta - \gamma) + \delta$.

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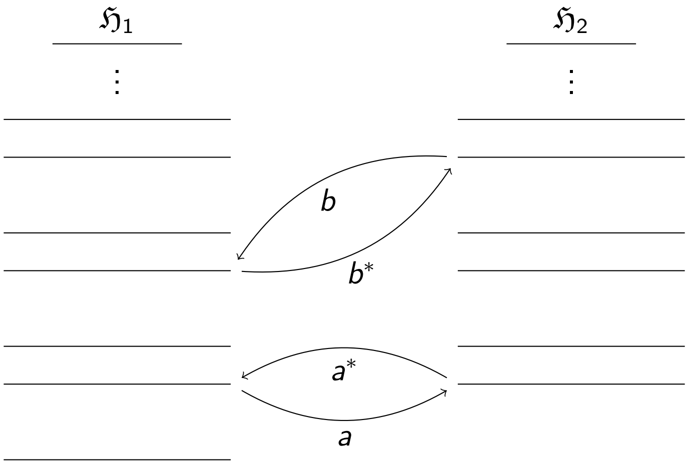

Here the eigenfunctions are eigenfunctions of a^*a and aa^*.

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The $L^2(\mathbb{R}, dx)$ representation of the oscillator algebra arises as a special case by taking $n = 1$ in which case $b = a$, $\gamma = -1$, and $\delta = 1$.

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For $\mathfrak{F}_1^{(n)}$, the closure is taken with respect to the norm generated from the inner product with measure $\rho_1^{(n)}(z, \bar{z}) = \lambda_n (z\bar{z})^{n-\frac{1}{2}} K_{1-\frac{1}{2n}}\left(\frac{(z\bar{z})^n}{n}\right)$.

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With respect to the above measures, it is clear that z^m and z^n are orthogonal if $n \neq m$ since the measures are radial.

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Let $\mathcal{B}_1^{(n)} : L^2(\mathbb{R}, dx) \to \mathfrak{F}_1^{(n)}$ and $\mathcal{B}_2^{(n)} : L^2(\mathbb{R}, dx) \to \mathfrak{F}_2^{(n)}$ denote the as-of-yet defined Bargmann transforms. The Bargmann transforms can be defined by the following commutative diagrams.

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When $n = 1$, these reduce exactly to the usual Bargmann transform associated to the oscillator algebra.

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$$f(z) = \int_{\mathbb{C}} f(w)\overline{K_1^{(n)}(z;w)}\, d\rho_1^{(n)}(w,\bar{w})$$

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The Bargmann transform is also effectively a short-time Fourier transform with Gaussian window, i.e. a time-frequency transform. Leaving the Fourier setting, short-time transforms become somewhat mysterious and non-unique. The coupled SUSY Bargmann transforms may be yet another short-time transform, distinct from one Williams, *et al* posited previously.

Further Questions

What happens if one takes non-integer n? Do integrals over $\mathbb{C}$ need to be replaced by integrals over infinite sectors? Can unitarity of the coupled SUSY Bargmann transforms still be attained?

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Are there peculiarities of the Fock spaces associated to coupled SUSYs? What happens if one combines Fock spaces for different values of n?

Thank you