Universal block tridiagonalization in B(H) and beyond,

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We prove every B(H) operator on a separable infinite dimensional complex Hilbert space has a basis for which its matrix is finite block tridiagonal, each with the same fixed precise block sizes given in a simple exponential form. An extension to unbounded operators occurs when a certain domain of definition condition is satisfied. And an extension to finite collections of operators holds, each finite collection with the same block sizes of larger exponential growth depending on the number of operators.

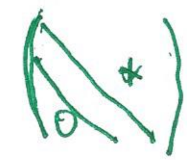
What is B(H) operator matrix sparcification?

Examples

∃ a basis for which

Spectral Th. Compact normal operator → diagonal matrix

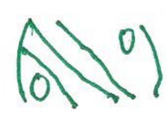
Operator with cyclic vector → upper Hessenberg



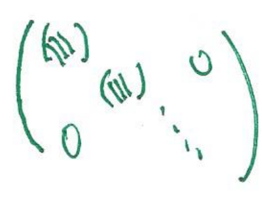
Pf Gram-Schmidt f, Tf, T^2f, T^3f, …
Must be linearly independent, and after Gram-Schmidt, every new vector is a linear combination of itself with the previous orthogonal vectors.

And observe, applying T to each T^nf and its predecessors lies in the span of T^{n+1}f and its predecessors.
And so also with the new orthogonal vectors.

Every operator → upper Hessenberg like (on direct sum H)
Pf. Choose f_2 orthogonal to the T invar subspace span { e_1, Te_1, T^2e_1, T^3e_1, …} and take this difference from each of the vectors {e_2, Te_2, T^2e_2, T^3e_2, …}, and so on.
I.e., a more general kind of Gram-Schmidt process.



Consequences:
Selfadjoint operator with cyclic vector → tridiagonal matrix
Every selfadjoint operator → direct sum of tridiagonal matrices



First application:

A seminal 4x4 matrix commutator problem (1973-76)

and consequences starting with

Pearcy-Topping problem (1971): Is every trace class trace zero operator a commutator of Hilbert-Schmidt operators?

NO

$$AB\text{-}BA = D = \begin{pmatrix} 1 & & & 0 \\ & -1/3 & & \\ & & -1/3 & \\ 0 & & & -1/3 \end{pmatrix}, \begin{pmatrix} 1 & & 0 \\ & -1/2 & \\ 0 & & -1/2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Observe: Take trace norm, apply Δ-ineq & Holder, divide by 2,

$$1 \le \|A\|_2 \|B\|_2$$

Key question: Is 1 minimal for $\|A\|_2\|B\|_2$? Yes for -½ & -1.
Computer evidence (1973): $4/3 \le \|A\|_2\|B\|_2$ for -1/3.

Pf. (1976): Gram Schmidt $e_1, Ae_1, A^*e_1, e_2, e_3, e_4$ (spans $\mathbb{C}^4$)

In this basis, $A = \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$ and leaves invariant $\begin{pmatrix} 1 & & & 0 \\ & -1/3 & & \\ & & -1/3 & \\ 0 & & & -1/3 \end{pmatrix}$

Diagonal entries: 1 = 12 + 13 (Notation: 12 means $a_{12}b_{21}-b_{12}a_{21}$)
-1/3 = -12 + 23 + 24
-1/3 = -13 - 23 + 43

Adding gets -1/3 = -24 - 43

Subtract from 1st Eq gets **4/3** = $12 + 13 + 24 + 43 \le \Sigma_{\textbf{distinct}}|ab| \le_{\text{Holder}} \|A\|_2\|B\|_2$

Generalizing: A few day later (1976) + dinner & a lemma from Ken Davidson,

Infinite matrix $\begin{pmatrix} d & 0 & 0 & 0 \\ 0 & -d_1 & 0 & 0 \\ 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$ with $d_n \geq 0$, $d = \Sigma\, d_n$

= a finite linear combo of Hilbert-Schmidt operators

$\Leftarrow_{1973}\ {}_{1976}\!\Rightarrow\ \Sigma\,(\log n)\, d_n < \infty$ (Hence commutator counterexample $d_n = 1/n \log^2 n$)

Commutators 1976-2004: Joel Anderson, Nigel Kalton
Theorem. Dykema-Figiel-W-Wodzicki (Advances 2004):
Completely general characterization (roughly speaking)
For all two-sided B(H)-ideals *I,J*, the linear span of the *I,J*-commutators = the linear span of diagonal operators with the **Cesaro mean** of their diagonal sequences forming diagonal operators in the product ideal *IJ*.

Single commutator characterizations remain wide open.

For both the 1971 Pearcy-Topping Hilbert-Schmidt commutator problem, and their compact commutator problem

Major contributors: Joel Anderson (Every compact operator is a K(H), B(H) commutator. 1976) Technique: used very sparse finite block triadiagonal matrices with growth rate n.

2006 Davidson-Marcoux-Radjavi (later independently Patnaik-W) extended Anderson's model to broaden the class of solvable compact operators.

2019-2020 Patnaik-Petrovic-Weiss created a universal representation theory of finite block tridiagonal matrices for all operators to prove limitations of the Anderson model and its possible extensions.

Trivial **Hilbert-Schmidt approximation** by block tridiagonal matrices.

Every B(H) matrix (only need rows and columns ℓ^2)

= an arbitrarily small Hilbert-Schmidt perturbation of a block tridiagonal matrix.

Pf. Fix a first upper left k x k block. Choose a wide k x n upper right block

and tall n x k lower left block with n large enough so the Hilbert-Schmidt norm squared together is $< \varepsilon/2$. This forces the 2nd n x n central block, and do the same to that, and so on.

BUT not exact, and not universal (block sizes depend on matrix rows and columns)

Indeed, the operator algebraists we queried on exactness believe it is new. Although in their work, block tridiagonal approximation is a common tool. But in commutator work, especial single commutator work, exactness seems essential.

Universal block tridiagonal forms from Staircase forms

But first

Staircase forms (generalizing the 4x4 sparcification method)

A = any B(H) matrix (or more generally, all words W(A, A*) e_n are defined)

Key spanning list: a self referencing sequence

$e_1, Ae_1, A^*e_1, e_2, A^2e_1, A^*Ae_1, e_3, AA^*e_1, A^{*2}e_1, e_4, Ae_2, A^*e_2, e_5, \ldots$

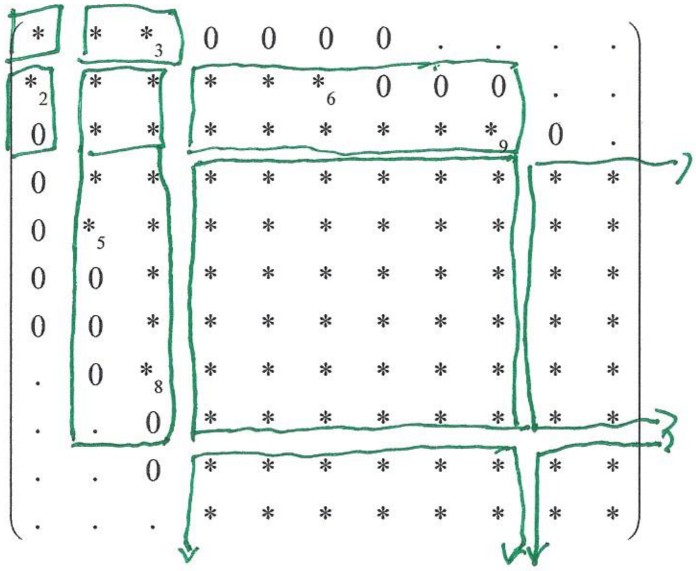
(Assume no collapsing. Collapsing can be handled with induction.)

After Gram-Schmidt, A = the staircase matrix 3-6-9

$$\begin{pmatrix}
* & * & *_3 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
*_2 & * & * & * & * & *_6 & 0 & 0 & 0 & \cdot & \cdot \\
0 & * & * & * & * & * & * & * & *_9 & 0 & \cdot \\
0 & * & * & * & * & * & * & * & * & * & * \\
0 & *_5 & * & * & * & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * & * & * & * & * \\
\cdot & 0 & *_8 & * & * & * & * & * & * & * & * \\
\cdot & \cdot & 0 & * & * & * & * & * & * & * & * \\
\cdot & \cdot & 0 & * & * & * & * & * & * & * & * \\
\cdot & \cdot & \cdot & * & * & * & * & * & * & * & *
\end{pmatrix}$$

Theorem. The same staircase form holds simultaneously for every finite collection of N operators. But the row/column support lengths are instead n(2N+1).

Universal block tridiagonal forms



The formula for these diagonal block sizes is:

1×1, and for $k \geq 2$, $2(3^{k-2}) \times 2(3^{k-2})$ square matrix
(I.e., 1×1, 2×2, 6×6, 18×18, ...)

And the off diagonal block sizes are obvious from these.
E.g., the upper diagonal rectangular blocks match up with the central block to its left, and the next central block below.

Theorem. For a finite collection of N operators, the same block sizes hold simultaneously and depend on N. They're larger and can be computed.

There are further sparcifications to be had.
And several open questions. See paper.

Thank you for listening.