Quantum Graphs and Quantum Graph C^*-Algebras

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Motivation

Graph theory has many interesting connections to operator algebras.

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- E.g., subfactors, group actions, representation theory, ...
- E.g., the theory of **Graph C^*-algebras**.

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Today's Plan: We want to **quantize** graph theory and the graph C^*-algebra construction.

1. Introduce **Quantum Graphs**.
2. Introduce **Quantum Graph C^*-algebras**.

Joint work with Kari Eifler, Christian Voigt, and Moritz Weber

Graphs

A (finite, directed) graph is a tuple $\mathcal{G} = (V, E, s, t)$ where

- V, E are finite sets (vertices and edges, respectively)
- $s, t : E \to V$ are functions (source and target maps).

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- $s, t : E \to V$ are functions (source and target maps).
- Thus each edge $e \in E$ is thus associated to an ordered pair $(s(e), t(e)) \in V \times V$. $s(e)$ is the source of e and $t(e)$ is the target of e.
- We allow graphs $\mathcal{G}$ to have multiple edges: The map $E \to V \times V; e \mapsto (s(e), t(e))$ need not be injective.

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- We allow graphs $\mathcal{G}$ to have multiple edges: The map $E \to V \times V; e \mapsto (s(e), t(e))$ need not be injective.
- If $\mathcal{G}$ has **no multiple edges**, then we simply identify $E \subseteq V \times V$. In this case we can form the adjacency matrix $A_G \in M_{V \times V}(\{0, 1\})$ of $\mathcal{G}$:

$$A_{\mathcal{G}}(v, w) = 1 \iff (v, w) \in E.$$

Then $\mathcal{G}$ is encoded by the pair $(V, A_{\mathcal{G}})$.

From Graphs to Quantum Graphs: NCG Perspective

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- **Our Problem**: **What is the quantum analogue of a graph**?

From Graphs to Quantum Graphs: NCG Perspective

- Non-commutative Geometry (NCG) is all about finding **quantum** (i.e., non-commutative) generalizations of familiar topological/algebraic/geometric structures: Quantum Topology, Quantum Probability, Quantum Groups, Spectral Triples, K-theory etc.
- **Our Problem**: **What is the quantum analogue of a graph**?
- This problem has been studied from many perspectives in QIT, NCG, and Quantum Algebra: [Duan-Severini-Winter 2010], [Weaver 2010], [Stahlke 2015], [Musto-Reutter-Verdon 2017], [BCEHPSW 2018], [Verdon 2020].
- At the surface, the various approaches to quantum graphs look quite different (e.g. C^*-algebras vs. operator spaces vs. Frobenius algebras))...but they ultimately lead to the same non-commutative theory.
- **Today**: We will focus on a C^*-algebraic description of quantum graphs [à la Musto-Reutter-Verdon].

Quantizing Graphs

Input: A finite directed graph $\mathcal{G} = (V, A_{\mathcal{G}})$ with $|V| = n$ vertices and no multiple edges.

Output: A C*-algebraic description of $\mathcal{G}$ that can be made non-commutative!

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- Let $B = C(V) = C^*\big((p_v)_{v\in V} \mid p_v^* = p_v^2 = p_v, \sum_v p_v = 1\big)$, the C*-algebra of functions on V.
- Equip B with the canonical tracial state $\psi : B \to \mathbb{C}$ given by the uniform probability on V. ψ is "**canonical**" because

  $$\psi = \mathrm{tr}_{\mathrm{End}(B)}\Big|_B \quad \text{where } B \hookrightarrow \mathrm{End}(B) = \text{ left-regular representation.}$$

- Use GNS inner product $\langle f|g\rangle = \psi(f^*g)$ to turn B into a Hilbert space.

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- Use GNS inner product $\langle f|g\rangle = \psi(f^*g)$ to turn B into a Hilbert space.
- View the adjacency matrix $A_{\mathcal{G}}$ as a linear map $A_{\mathcal{G}} \in \mathrm{End}(B)$ in the obvious way:

$$A_{\mathcal{G}} p_w = \sum_{w \in V} A_{\mathcal{G}}(v, w) p_v.$$

Quantizing Adjacency Matrices

So far:

$$\mathcal{G} \rightsquigarrow \Big(B = C(V), \psi = \mathrm{tr}_{\mathrm{End}(B)}\Big|_B, A_{\mathcal{G}} \in \mathrm{End}(B)\Big).$$

Problem: How to intrinsically capture the fact that $A_{\mathcal{G}}$ is an **adjacency matrix**?

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Problem: How to intrinsically capture the fact that $A_\mathcal{G}$ is an **adjacency matrix**?

Answer: $A_\mathcal{G} \in M_n(\{0,1\}) \iff A_\mathcal{G}$ is **idempotent** with respect to **Schur multiplication** in End(B)!

- Let $m : B \otimes B \to B$ be the algebra multiplication, $m^* : B \to B \otimes B$ its Hilbertian adjoint.

$$m(p_v \otimes p_w) = \delta_{v,w} p_v, \qquad m^*(p_v) = (\dim B)(p_v \otimes p_v).$$

- Simple calculation: Given $X, Y \in \mathsf{End}(B)$,

$$\text{Schur Multiplication} : X \odot Y = \frac{1}{\dim B} m(X \otimes Y) m^*$$

Conclusion: $A_\mathcal{G} \in M_n(\{0,1\}) \iff \frac{1}{\dim B} m(A_\mathcal{G} \otimes A_\mathcal{G}) m^* = A_\mathcal{G}.$

Quantum Graphs

We now go non-commutative! Let B be a finite-dimensional C*-algebra equipped with its canonical tracial state

$$\psi : B \to \mathbb{C}; \quad \psi = \mathrm{tr}_{\mathrm{End}(B)}\Big|_B$$

Definition

A linear map $A \in \mathrm{End}(B)$ is called a quantum adjacency matrix if

$$\frac{1}{\dim B} m(A \otimes A) m^* = A \quad (A \text{ is a "}\textbf{quantum Schur idempotent}\text{"})$$

We call the triple $\mathcal{G} = (B, \psi, A)$ a quantum graph.

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- Classical graphs $\iff$ quantum graphs with **abelian** B.
- For certain applications, we can consider quantum graphs $\mathcal{G} = (B, \psi, A)$ with **non-tracial states** ψ...but not today.

Some Basic Examples

Fix any pair (B, ψ).

1. The complete quantum graph over B is $\mathcal{K}(B) = (B, \psi, A)$,

$$A = (\dim B)\psi(\cdot)1_B \qquad \textbf{Classical case: } A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & 1 & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

2. The trivial quantum graph over B is $\mathcal{T}(B) = (B, \psi, A)$ where

$$A = \mathrm{id}_{B \to B} \qquad \textbf{Classical case: } A = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

3. If $B = \bigoplus_k M_{n(k)}(\mathbb{C})$ and $d = (d^{(k)})_k \in \bigoplus_k D_{n(k)} \subset B$ is **diagonal** with $\mathbf{tr}_{n(k)}(d^{(k)}) = 1$ for all k, then

$$A_d(x) = dx \qquad \text{is a quantum adjacency matrix.}$$

We call $\mathcal{D}(B) = (B, \psi, A_d)$ a diagonal quantum graph over B.

More Examples: Operator Space Picture

Picking interesting quantum adjacency matrices out of thin air is not so easy.

Alternate Approach [Weaver]: Given a f.d. C*-algebra B and a faithful embedding $B \subseteq B(H)$, we can look at B'-B'-**bimodules**

$$_{B'}S_{B'} \subseteq B(H).$$

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Write $S = P(B(H))$ for some **bimodule projection**

$$P \in CB_{B'-B'}(B(H)) \cong B \otimes B^{\mathsf{op}}$$

$$\left\{T \mapsto \sum_i a_i T b_i\right\} \longleftrightarrow \left\{P = \sum_i a_i \otimes b_i^{\mathsf{op}} \in B \otimes B^{\mathsf{op}}\right\}$$

Fact 1: Any $P \in B \otimes B^{\mathsf{op}}$ has a "**Choi-Jamoiłkowski**" form:

$$\exists! A \in \mathsf{End}(B) \text{ so that } P = P_A = \frac{1}{\dim B}(1 \otimes A)m^*(1_B).$$

Fact 2: $P_A^2 = P_A \iff$ **A is a quantum adjacency matrix**!

Conclusion: Quantum graphs $\mathcal{G} = (B, \psi, A) \longleftrightarrow$ operator spaces $S \subseteq B(H)$ that are B'-B'-bimodules.

Quantum Graphs in “Nature”

- **[Weaver 2010]**: The bimodules ${}_{B'}S_{B'} \subseteq B(H)$ arise as **quantum relations on B**. If $B = C(V) \subset B(\ell^2(V))$, this recovers ordinary relations (=graphs) on V.

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- **[Duan-Severini-Winter 2010, Stahlke 2015]**: In QIT, considered quantum channels

$$\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C}).$$

Zero-error capacity of Φ: $\rightsquigarrow$ study non-commutative confusability graph of Φ: $S_\Phi \subset M_n(\mathbb{C})$ is an operator system.

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- **[Musto-Reutter-Verdon, BCEHPSW]** C*-algebraic quantum graphs $\mathcal{G} = (B, \psi, A)$ appear in the theory of non-local games (graph isomorphism games), quantum teleportation and superdense coding schemes in QIT, representation theory of quantum symmetry groups of graphs, ...

C*-algebras associated to graphs

Let $\mathcal{G} = (V, E, s, t)$ be a finite graph.

The graph C*-algebra $C^*(\mathcal{G})$ is the universal C*-algebra generated by projections $(p_v)_{v\in V}$ and partial isometries $(s_e)_{e\in E}$ satisfying

- $s_e^* s_f = \delta_{e,f} p_{t(e)}$, $(e \in E)$.
- $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever v is **not a sink** $(s^{-1}(v) \neq \emptyset)$.

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A Cuntz-Krieger Algebra is a graph C*-algebra of a graph $\mathcal{G}$ with no sources or sinks. A Cuntz-Krieger algebra is generated by partial isometries $(s_e)_{e \in E}$ subject to the relations

$$s_e^* s_e = \sum_{f \in E} \hat{A}(e,f) s_f s_f^* \qquad (e \in E).$$

where $\hat{A} = [\hat{A}(e,f)]_{e,f \in E} = [\delta_{t(e),s(f)}]$ is the edge matrix of $\mathcal{G}$.

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Goal: Generalize Cuntz-Krieger Algebras to Quantum Graphs!

Cuntz-Krieger Algebras and the Line Graph

Given a finite graph $\mathcal{G} = (V, E, s, t)$ without sources/sinks, we can form the line graph $L\mathcal{G}$ of $\mathcal{G}$:

- $V(L\mathcal{G}) = E$
- $E(L\mathcal{G}) = \{(e, f) \in E \times E : t(e) = s(f).\}$.

$L\mathcal{G}$ is always **without multiple edges** with adjecency matrix $A_{L\mathcal{G}}$ given by $A_{L\mathcal{G}} = \hat{A}$, the **edge matrix** of $\mathcal{G}$.

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Fact: The Cuntz-Krieger Algebra $C^*(\mathcal{G})$ depends only on the line graph $L\mathcal{G}$.

$$\text{Notation: } C^*(\mathcal{G}) = \mathcal{O}_{L\mathcal{G}}$$

We can obviously associate a CK-algebra $\mathcal{O}_\mathcal{G}$ to graph $\mathcal{G} = (V, A_\mathcal{G})$ without multiple edges.

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Examples:

1. $\mathcal{G} = \{1 \text{ vertex}, n \text{ loops}\}$, $L\mathcal{G} = K_n$, $C^*(\mathcal{G}) = \mathcal{O}_{K_n} = \mathcal{O}_n$ (Cuntz Algebra!)
2. $\mathcal{G} = T_n$ (trivial graph on n vertices, with self-loops), $LT_n = T_n$, $C^*(T_n) = \mathcal{O}_{T_n} = C(S^1)^{\oplus n}$.

Properties of Graph C*-algebras

- $C^*(\mathcal{G}) \neq 0$.
- $C^*(\mathcal{G})$ is always nuclear.
- $C^*(\mathcal{G})$ is always unital (for finite $\mathcal{G}$).
- Graph C*-algebras are amenable to classification
- Include many naturally occuring examples of C*-algebras.
- Have nice guage actions, guage/Cuntz-Krieger uniqueness theorems, ...

Quantum Cuntz-Krieger Algebras

Now let $\mathcal{G} = (B, \psi, A)$ be a quantum graph.

Definition

The quantum Cuntz-Krieger Algebra associated to $\mathcal{G}$ is the universal C*-algebra $\mathcal{O}_{\mathcal{G}}$ generated by the range of a linear map $S : B \to \mathcal{O}_{\mathcal{G}}$ satisfying the relations

1. $\mu(1 \otimes \mu)(S \otimes S^* \otimes S)(1 \otimes m^*)m^* = S$
2. $\mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A$

where $\mu : \mathcal{O}_{\mathcal{G}} \otimes \mathcal{O}_{\mathcal{G}} \to \mathcal{O}_{\mathcal{G}}$ is the multiplication map and $S^*(b) = S(b^*)^*$.

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Why is this a reasonable definition?

Theorem

If $\mathcal{G}$ is classical (i.e., B is abelian), then $\mathcal{O}_{\mathcal{G}}$ is the usual Cuntz-Krieger algebra associated to $\mathcal{G}$.

Quantum CK-Algebras: Unpacking the Definition

$$\mathcal{G} = (B, \psi, A) \rightsquigarrow \mathcal{O}_{\mathcal{G}} = C^*\left((S(b))_{b \in B} \;\middle|\; \substack{S: B \to \mathcal{O}_{\mathcal{G}} \text{ linear} \\ \mu(1 \otimes \mu)(S \otimes S^* \otimes S)(1 \otimes m^*)m^* = S \\ \mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A} \right)$$

Write $B = \bigoplus_{k=1}^{m} M_{n(k)}(\mathbb{C})$ with standard matrix unit basis $\{e_{ij}^{(k)}\}$.
For $1 \leq k \leq m$, put

$$S^{(k)} = \frac{\dim B}{n(k)} [S(e_{ij}^{(k)})] \in M_{n(k)}(\mathcal{O}_{\mathcal{G}}).$$

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Then $\mu(1 \otimes \mu)(S \otimes S^* \otimes S)(1 \otimes m^*)m^* = S$

$\iff S^{(k)} {S^{(k)}}^* S^{(k)} = S^{(k)}, \quad k = 1, \ldots, m$

$\iff$ each $S^{(k)}$ is a **partial isometry** in $M_{n(k)}(\mathcal{O}_{\mathcal{G}})$.

$$\mu(S^* \otimes S)m^* = \mu(S \otimes S^*)m^* A$$

$$\iff {S^{(k)}}^* S^{(k)} = (S^{(1)} {S^{(1)}}^*, \ldots, S^{(m)} {S^{(m)}}^*) \tilde{A}$$

where $\tilde{A}_{ijk}^{xyz} = \frac{n(z)}{n(k)} A_{ijk}^{xyz}$ (reweighted adjacency matrix).

Quantum CK-Algebras: Unpacking the Definition

Summary: Given $\mathcal{G} = (B, \psi, A)$, $B = \bigoplus_{k=1}^{m} M_{n(k)}$, the quantum Cuntz-Krieger Algebra $\mathcal{O}_{\mathcal{G}}$ is:

1. Generated by the coefficients of m **partial isometries** $S^{(k)} \in M_{n(k)}(\mathcal{O}_{\mathcal{G}})$,
2. Subject to the **quantum CK relations**

$$S^{(k)^*} S^{(k)} = (S^{(1)} S^{(1)^*}, \ldots, S^{(m)} S^{(m)^*}) \tilde{A}.$$

Basic Examples, Basic Properties

Example 1: Complete Quantum Graph $\mathcal{G} = \mathcal{K}(M_n)$.

$$\mathcal{O}_{\mathcal{K}(M_n)} = C^*\left(S_{ij},\ 1 \leq i,j \leq n \mid \substack{S=[S_{ij}] \text{ is a partial isometry} \\ \sum_l S_{li}^* S_{lj} = \delta_{i,j} n \sum_{x,l} S_{xl} S_{xl}^*}\right)$$

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- We have a quotient map

$$\mathcal{O}_{\mathcal{K}(M_n)} \to \mathcal{O}_{n^2} = \mathcal{O}_{K_{n^2}} \quad \text{(the Cuntz algebra)}$$
$$S_{ij} \mapsto n^{-1/2} \hat{S}_{ij} \quad (\hat{S}_{ij} = \text{standard Cuntz isometry}).$$

- Conjecture: $\mathcal{O}_{\mathcal{K}(M_n)}$ and $\mathcal{O}_{n^2}$ should be "closely related". E.g., there ought to be embeddings

$$\mathcal{O}_{n^2} \hookrightarrow M_{n^2}(\mathcal{O}_{\mathcal{K}(M_n)}), \quad \mathcal{O}_{\mathcal{K}(M_n)} \hookrightarrow M_{n^2}(\mathcal{O}_{n^2}).$$

...work in progress.

Basic Examples, Basic Properties

Example 2: Trivial Quantum Graph $\mathcal{G} = \mathcal{T}(M_n)$.

$$\mathcal{O}_{\mathcal{T}(M_n)} = C^*\left(S_{ij},\ 1 \leq i,j \leq n \mid {}^{S=[S_{ij}] \text{ is a partial isometry}}_{\sum_l S_{li}^* S_{lj} = \sum_l S_{il} S_{jl}^*}\right)$$

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- $\mathcal{O}_{\mathcal{T}(M_n)}$ is **non-unital**, **non-nuclear**.
- We have quotient maps

$$\begin{aligned} \mathcal{O}_{\mathcal{T}(M_n)} &\to C(S^1)^{\star n} \quad \text{(non-unital free product)} \\ S_{ij} &\mapsto \delta_{ij} z_i. \\ \mathcal{O}_{\mathcal{T}(M_n)} &\to U^{nc}(n) \\ S_{ij} &\mapsto u_{ij}. \end{aligned}$$

where $U^{(nc)}(n) = C^*\left(u_{ij},\ 1 \leq i,j \leq n \;\middle|\; U = [u_{ij}] \text{ unitary}\right)$ is Brown's Universal noncommutative unitary algebra.

- $M_n(O^+_{\mathcal{T}(M_n)}) \cong M_n \star (C(S^1) \oplus \mathbb{C})$.

Basic Examples, Basic Properties

Example 3: Diagonal Quantum Graph $\mathcal{G} = \mathcal{D}(M_2)$
$A_d(e_{ij}) = \delta_{i,1} 2e_{ij}$.

$$\mathcal{O}_{\mathcal{D}(M_2)} = C^*\left(S_{ij},\ 1 \leq i, j \leq 2 \mid {}^{S=[S_{ij}] \text{ is a partial isometry}}_{\sum_l S_{li}^* S_{lj} = \delta_{\{i=j=1\}} \sum_l 2S_{1l}S_{1l}^*}\right)$$

- Get $S_{l2}^* S_{l2} = 0 \implies S_{l2} = 0$.
- Thus, the canonical map $S : B \to \mathcal{O}_{\mathcal{G}}$ need not be injective!

Conclusion: Quantum Cuntz-Krieger Algebras seem to some new and interesting properties! Lots of work to be done!

THANKS FOR LISTENING.

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