Distributions of functions in noncommuting random variables

Serban T. Belinschi

CNRS - Institut de Mathématiques de Toulouse

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1. Noncommutative distributions
   - Noncommutative (joint) distributions
   - Analytic transforms of noncommutative distributions
2. Applications
   - Distributions of polynomials and analytic functions in noncommuting variables
   - Freeness

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Noncommutative probability

The process of passing from "commutative" to "noncommutative" `[insert object here]` is (most) often done by switching the perspective from the `[object]` to some algebra of functions defined on the `[object]`, and trying to eliminate the commutativity assumption on that algebra.

Noncommutative probability spaces generalize $(L^\infty([0,1],\mathrm{d}x),\mathbb{E}[\,\cdot\,]=\int\cdot\,\mathrm{d}x)$.
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noncommutative probability space $=$ von Neumann algebra with state.

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Noncommutative (joint) distributions

Classical distribution on $\mathbb{R}^n$ = linear functional, continuous on some space of test functions on $\mathbb{R}^n$.

Our class of noncommutative test functions is $\mathbb{C}\langle X_1, \ldots, X_n\rangle$, the algebra of polynomials in n selfadjoint noncommuting indeterminates[1] (so $X_1, X_2, \ldots, X_n$ satisfy no algebraic relation)

1. A *noncommutative distribution* is a linear $\mu\colon \mathbb{C}\langle X_1, \ldots, X_n\rangle \to \mathbb{C}$ such that $\mu(1) = 1$;
2. μ is *positive* if $\mu(P^*P) \geq 0$ for all $P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$;
3. μ is *bounded* if for any $P \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$ there is an $R_P > 0$ such that $\mu((P^*P)^k) < R_P^{2k}$ for all $k \in \mathbb{N}$;
4. μ is *tracial* if $\mu(PQ) = \mu(QP)$ for any $P, Q \in \mathbb{C}\langle X_1, \ldots, X_n\rangle$.

The set of positive, bounded tracial distributions is denoted by Σ_0.

[1] For the rest of the talk, think $n = 2$!

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Realizing and encoding nc distributions

As in classical probability: one can realize a given distribution $\mu \in \Sigma_0$ as the distribution of a tuple of *selfadjoint elements* ("random variables") $\mathbf{x} = (x_1, \ldots, x_n)$ in a tracial C^*-algebra, here via the GNS construction with respect to $\langle P, Q\rangle_\mu = \mu(Q^* P)$. We write $\mu_{\mathbf{x}}$ when we view μ as the distribution of the variables $\mathbf{x} = (x_1, \ldots, x_n)$

Convention: Upper case X_j denote indeterminates, lower case x_j denote random variables in a tracial C^*- or W^*-algebra.

By linearity, the *matrix of moments* (or moment matrix) $M(\mu)$ given by

$M(\mu)_{v,w} = \mu\left((X^w)^* X^v\right), \ v, w \in \mathbb{F}_n^+,$ the free semigroup in n generators,

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Free noncommutative transforms (nc transforms)

Cauchy transform $G_\mu(z_1,\ldots,z_n)=\int_{\mathbb{R}^n}\prod_{j=1}^n(z_j-t_j)^{-1}\,\mathrm{d}\mu(t_1,\ldots,t_n)$ encodes the *classical* distribution μ (see, for instance, Koranyi);

Nc Cauchy transform[2] encodes the *noncommutative* distribution μ: Let $\mathbf{x}=\mathrm{diag}(x_1,\ldots,x_n)$, $G_{\mu,1}(b)=(\mu\otimes\mathrm{id}_{\mathbb{C}^{n\times n}})\left[(b-\mathbf{x})^{-1}\right]$. Amplify:

$$G_{\mu,m}(b)=(\mu\otimes\mathrm{id}_{\mathbb{C}^{mn\times mn}})\left[(b-\mathbf{x}\otimes I_m)^{-1}\right],\quad m\in\mathbb{N},b\in\mathbb{C}^{mn\times mn}.$$

$G_{\mu,m}(b^{-1})$ extends to a nbhd of 0 as $(\mu\otimes\mathrm{id}_{\mathbb{C}^{mn\times mn}})\left[b(1-(\mathbf{x}\otimes I_m)b)^{-1}\right]$. By choosing an appropriate b (an upper diagonal $m\times m$ matrix of $n\times n$ permutation matrices will do), the expansion of $G_{\mu,m}(b^{-1})$ yields any entry $M(\mu)_{v,w},|v|+|w|\leq m-2$.

It extends analytically as an nc function to the nc upper half-plane $H^+=\{b\colon \Im b>0\},-G_\mu(H^+)\subseteq H^+$. (See Voiculescu, Popa-Vinnikov.)

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Aside: Approximation of nc transforms

Nc distributions $\{\mu_k\}_{k\in\mathbb{N}} \subset \Sigma_0$ converge to distribution $\mu \in \Sigma_0$ if and only if the *nc Cauchy transforms* $G_{\mu_k} \to G_\mu$ as $k \to \infty$.

Denote

$$\Sigma_0^{\text{fin}} = \{\mu \in \Sigma_0 \colon \text{there exists } d \in \mathbb{N}, x_1, \dots x_n \in \mathbb{C}^{d\times d} \text{ such that } \mu(P) = \text{tr}_d(P(x_1, \dots, x_d)) \text{ for all } P \in \mathbb{C}\langle X_1, \dots, X_n\rangle\}.$$

Stating that $\{G_\mu \colon \mu \in \Sigma_0^{\text{fin}}\}$ is dense in the space of nc functions that map H^+ to $H^- = -H^+$ and vanish at infinity with residue one[3] is equivalent to stating that all bounded positive tracial distributions have microstates (see J. Williams), which we now know to be false.

Contrast that with the fact that classical distributions are approximable by atomic ones, corresponding to functions of the type $(z_1, \dots, z_n) \mapsto \sum_{i=1}^N \alpha_i \prod_{j=1}^n \frac{1}{z_j - s_j^{(i)}}, N \in \mathbb{N}, \alpha_i \geq 0, \sum_{i=1}^N \alpha_j = 1, s_j^{(i)} \in \mathbb{R}$.

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Extending the nc Cauchy transform I

Observe: $\mathbf{x} = \operatorname{diag}(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i \otimes e_{i,i}$. Allowing instead

$$\mathbf{x} = \sum_{j=1}^{n} x_j \otimes c_j - 1 \otimes c_0, \quad c_j = c_j^* \in \mathbb{C}^{d\times d}, d \in \mathbb{N}, \tag{1}$$

allows for explicit computations of $\mu(P)$ for arbitrary nc polynomials, or even rational functions, P (realization/linearization of P).

Thus, from now on,

$$G_{\mu_{\mathbf{x}}}(b) = (\mu \otimes \mathrm{id}_{\mathbb{C}^{d\times d}}) \left[(1 \otimes b - \mathbf{x})^{-1} \right]$$
$$= (\mu \otimes \mathrm{id}_{\mathbb{C}^{d\times d}}) \left[\left(1 \otimes (b + c_0) - \sum_{j=1}^{n} x_j \otimes c_j \right)^{-1} \right].$$

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Extending the nc Cauchy transform II

Allowing further

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allows for “explicit” computations of $\mu(f)$ for nc analytic functions f in variables $x_1, \ldots, x_n$.

If f is an entire analytic function, then realization (2) can be done with compacts $c_1, \ldots, c_n \in B(\ell^2(\mathbb{N}))$, and thus convergence of $(1 \otimes p_j)(\mathbf{x} + 1 \otimes c_0)(1 \otimes p_j) \to \mathbf{x} + 1 \otimes c_0$ as $j \to \infty$ is in norm (p_j is the projection on $\mathrm{span}\{1, \ldots, j\} \subset B(\ell^2(\mathbb{N}))$).

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Freeness via analytic nc functions

As nc Cauchy transforms characterize nc distributions, any form of *independence* must be describable via (some modification of) nc Cauchy transforms.

Voiculescu's *free independence* (or *freeness*) has the following characterization in terms of nc Cauchy transforms (2000):

Definition/Theorem (Voiculescu)

Tuples $(x_1, \ldots, x_n)$ *and* $(y_1, \ldots, y_n)$ *are free iff there exist nc self-maps* ω_1, ω_2 *of* H^+ *such that*

$$\begin{aligned}(\omega_1(b) + \omega_2(b) - b)^{-1} \\ = \quad G_{\mu_{\mathbf{x}+\mathbf{y}}}(b) = G_{\mu_{\mathbf{x}}}(\omega_1(b)) = G_{\mu_{\mathbf{y}}}(\omega_2(b)),\end{aligned}$$

as an equality of nc maps.

(*Moreover,*

$$E_{\mathbf{x}}\left[(b - \mathbf{x} - \mathbf{y})^{-1}\right] = (\omega_1(b) - \mathbf{x})^{-1}, \tag{3}$$

and the same for $\mathbf{y}$.)

Here $\mathbf{x}, \mathbf{y}$ *should be understood in the sense of Equation* (1)*!*

Atoms of polynomials in free variables I

Consider the case $n = 1$ in Voiculescu's Definition/Theorem, and let $P = P^*$ be polynomial in two noncommuting indeterminates.

Question: Under what conditions on x_1, y_1, P is it possible that $\ker P(x_1, y_1) \neq \{0\}$?

Many negative answers, starting with Shlyakhtenko-Skoufranis (2013).

Atoms of polynomials in free variables I

Consider the case $n = 1$ in Voiculescu's Definition/Theorem, and let $P = P^*$ be polynomial in two noncommuting indeterminates.

Question: Under what conditions on x_1, y_1, P is it possible that $\ker P(x_1, y_1) \neq \{0\}$?

Many negative answers, starting with Shlyakhtenko-Skoufranis (2013).

Atoms of polynomials in free variables II

Question: Under what conditions $\ker P(x_1, y_1) \neq \{0\}$?
We answer this question in joint work with H. Bercovici and W. Liu (2019), in two steps.

1. We find a realization
$$L(x_1, y_1) = x_1 \otimes c_1 + y_1 \otimes c_2 - 1 \otimes c_0, \quad c_j \in \mathbb{C}^{d \times d},$$
such that
$$\ker(A \otimes e_{1,1} + L(x_1, y_1)) \overset{\text{MvN}}{\sim} \ker(A - P(x_1, y_1)) \oplus (1 \otimes 0_{d-1});$$
2. We use the Julia-Carathéodory derivative of the reciprocals of the Cauchy transforms of the distributions of $L(x_1, y_1), x_1 \otimes c_1, y_1 \otimes c_2$.

Part 2 involves several technical sub-steps. The answer is explicit in terms of the Julia-Carathéodory derivatives of ω_1, ω_2, which are in principle fully computable, via Voiculescu's relations (3).
The only drawback: with our methods, $d \in \mathbb{N}$ may be very large and the technical sub-steps quite involved.

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Atoms of polynomials in free variables III

Specifically, with the notation $\Pi = \ker(L(x_1, y_1))$, we know how to "safely cut" the expectation of this kernel left and right with projections $p_1, p_2 \in \mathbb{C}^{d\times d}$ so that the expectation of $\tilde{\Pi} = \ker \mathrm{diag}(p_1, p_2) \begin{bmatrix} 0 & L(x_1, y_1) \\ L(x_1, y_1) & 0 \end{bmatrix} \mathrm{diag}(p_1, p_2)$ is invertible in the reduced algebra $\mathrm{diag}(p_1, p_2)\mathbb{C}^{2d\times 2d}\mathrm{diag}(p_1, p_2)$. Voiculescu's Definition/Theorem still holds for the "cut" random variables, so we may apply (with the *new* ω_1, ω_2)

Theorem (B., Bercovici, Liu '19)

Under the above invertibility assumption,

$$\ker(\omega_1'(c_0)(1)^{-\frac{1}{2}}(x_1 \otimes c_1 - \omega_1(c_0))\omega_1'(c_0)(1)^{-\frac{1}{2}}) = E_{x_1}[\omega_1'(c_0)(1)^{\frac{1}{2}}\tilde{\Pi}\omega_1'(c_0)(1)^{\frac{1}{2}}]$$

and $\tau(\tilde{\Pi})+1=\tau(E_{x_1}[\omega_1'(c_0)(1)^{\frac{1}{2}}\tilde{\Pi}\omega_1'(c_0)(1)^{\frac{1}{2}}] + E_{y_1}[\omega_2'(c_0)(1)^{\frac{1}{2}}\tilde{\Pi}\omega_2'(c_0)(1)^{\frac{1}{2}}])$.

(Here τ is the trace on the reduced algebra.)

Entire nc functions in free variables

Joint work in progress with V. Vinnikov

We construct

$$L(x_1, y_1) = 1 \otimes c_0 + x_1 \otimes c_1 + y_1 \otimes c_2, \quad c_1, c_2 \in B(\ell^2(\mathbb{N})) \text{ compact;}$$

Approximation with finite-rank operators allows us to recover Voiculescu's result (3);

The Murray-von Neumann equivalence of projections $\ker(A \otimes e_{1,1} + L(x_1, y_1)) \overset{\text{MvN}}{\sim} \ker(A - f(x_1, y_1)) \oplus (1 \otimes 0_{d-1})$ still holds with d infinite;

The formulation of the condition for the existence of the kernel in terms of the Julia-Carathéodory derivatives of ω_1, ω_2 still holds, with some modifications;

However, the full extent of properties imposed upon x_1, y_1 by these conditions is not clear to us yet.

Thank you!

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