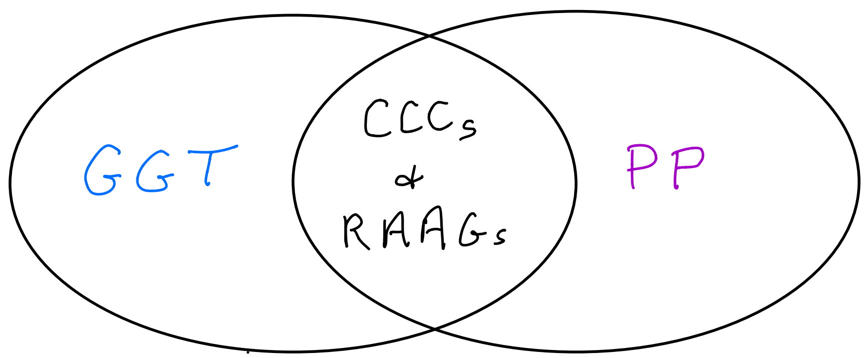
From Geometric Group Theory to Polyhedral Products



Lecture I: An Introduction to Geometric Gp Theory

Lecture II: CAT(0) Cube Complexes (CCCs) and Right-Angled Artin Groups (RAAGs)

Lecture III: Introduction to Polyhedral Products and Connections with Geometric Group Theory

Overview of Geometric Group Theory

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CAT(0) cube complexes and RAAGs

D. Wise, *From Riches to Raags: 3-Manifolds, Right-Angled Artin Groups, and Cubical Geometry.* CBMS Regional Conference Series in Math, 117.

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M. Bestvina, *Geometric group theory and 3-manifolds hand in hand: the fulfillment of Thurston's vision.* Bulletin of the AMS, vol 51, no 1, pp 53-70.

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Polyhedral products and quasi-median spaces

A. Bahri, M. Bendersky, and F. R. Cohen, *Polyhedral products and features of their homotopy theory.* arXiv:1903.04114

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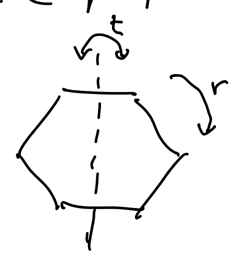
I. An Introduction to Geometric Group Theory

Various approaches to studying a group G

- algebraic: subgroups, quotient, torsion, decomposition, cohomology
- combinatorial: $G = \langle S \mid R \rangle = \langle \text{generators} \mid \text{relations} \rangle$
  find algorithms to answer question about G
  Word problem: Is there an algorithm to determine when two words in S represent the same element in G.

GGT
- geometric: find an action $G \curvearrowright X$ = metric space & use the geometry of X to deduce properties of G.

Eg: $D_{12} = \langle t, r \mid t^2 = 1,\ r^6 = 1,\ rt = tr^{-1} \rangle$



To get useful info about G, we need some conditions on X and on $G \curvearrowright X$.

<u>Some assumptions</u>:

- Always assume that X is a <u>geodesic metric space</u>: $\forall\, x, y \in X$
  $d(x,y)$ = length of $\underbrace{\text{the shortest path}}_{\text{"geodesic"}}$ from x to y
  (and X is proper, i.e. closed balls are compact).
- Always assume G acts by <u>isometries</u> on X (action is given by $G \to \mathrm{Isom}(X)$.)
- We prefer the action to be <u>geometric</u>

. .

geom action: 1) properly discont $\Rightarrow$ Stab(x) is finite
2) cocompact X/G is compact

Q Given G, does such an action always exist?

A Yes: X = Cayley graph of G = $\mathrm{Cay}(G, S)$ ← gen set

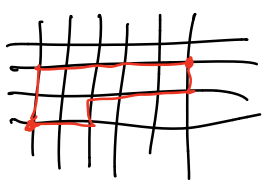
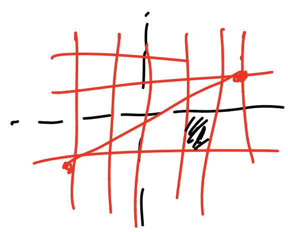
Vertices = G

Edges: $g_1 \xrightarrow{s} g_2$ $g_2 = g_1 s$, $s \in S$

This is proper providing S is finite.

Note: G may act geometrically on many different spaces X. Does it matter which one we choose?

No: Milnor-Svarc Lemma: If $G \curvearrowright X$ and $G \curvearrowright Y$ are two geometric actions, then X and Y are quasi-isometric.

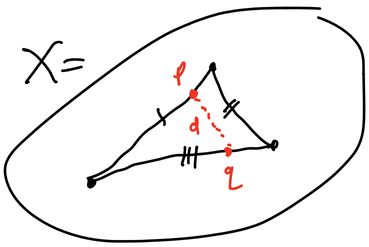
Eg: grid:  $\underset{\longleftarrow}{\hookrightarrow}$ $\mathbb{R}^2$ 

Yes: Some types of geometries are more useful than others. Want to find an action of G on a "useful" space X.

Q: What kind of spaces are useful?

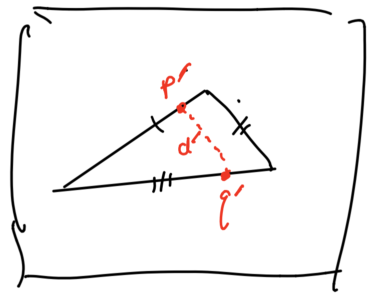
<u>CAT(0) spaces</u>: A geodesic metric space X is CAT(0) if geodesic triangles in X are at least as thin as comparison triangles in $\mathbb{R}^2$.

$X =$



$d \le d'$

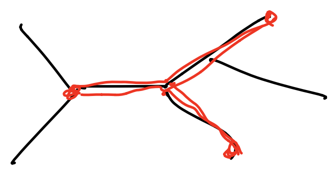
$\mathbb{R}^2$



A space which is locally CAT(0) is called "non-positively curved" NPC.

<u>Eg</u>

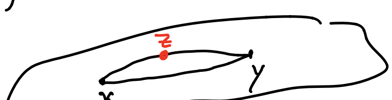
1) $X = \mathbb{R}^n$ Euclidean space
2) $X = \mathbb{H}^n$ hyperbolic space
3) M any Riemannian mnfld with non-positive sectional curvature, then M is NPC and $\tilde{M}$ is CAT(0).
4) $X =$ tree is CAT(0)



5) $X =$ Euclidean bldg
6) next time: CAT(0) cube complexes

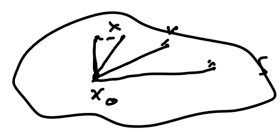
Properties of CAT(0) spaces: X

- geodesics in X are unique



comparison triangle

- X is contractible



X

- local geodesics in X are global geodesic
- NPC + simply conn $\Longrightarrow$ X globally CAT(0)
  (X NPC $\Rightarrow$ $\tilde{X}$ CAT(0))

Now say G is a group acting geometrically on a CAT(0) space X. (Call these <u>CAT(0) groups</u>)
What can we deduce about G.

*
- G is finitely presented
- G has solvable word problem & conj problem
- Structure subgps
  - solvable subgp are virtually abelian
  - all abelian subgps are fin gen

  ⋮

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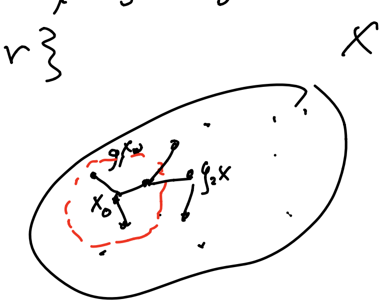
- if G is torsion-free then G is geometrically finite (i.e. it has a finite Eilenberg-MacLane space).

* Idea of how geom of X $\rightsquigarrow$ alg + comb properties of G

Choose $x_0 \in X$, map $G \longrightarrow X$ by $g \mapsto gx_0$
Set $S = \{g \in G \mid d(x_0, gx_0) < r\}$
action proper $\Longrightarrow$ S is finite
action cocompact $\Rightarrow$ for suff large r, S generates G
Get $Cay(G, S) \longrightarrow X$



* $G \curvearrowright X$ proper + cocompact.
proper $\Rightarrow$ $\mathrm{Stab}(x)$ finite $\Rightarrow$ $\mathrm{Stab}(x)=\{e\}$, action is <u>free</u>.
$\cancel{X}$ CAT(0) $\Rightarrow$ X contractible, so X/G has

$\pi_1(X/G) = G$ and universal covering space $X \simeq *$
and X/G is compact. Show $X/G \simeq \underbrace{\text{a}}_{\text{finite}}$ CW-complex.