Graph Products, Wreath Products, and Polyhedral Products in Geometric Group Theory

Lecture I: Cell Complexes, Cellular Homology, and Geometric Finiteness Conditions

Peter Kropholler

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Lecture I

PAIR = category of pairs of top spaces

PAIR*

$$(X, Y) \longrightarrow (X', Y')$$

$$X \xrightarrow{f} X'$$

$$f|_{Y}(Y) \subseteq Y'$$

$$H_n(X) := H_n(X, \emptyset).$$

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LES EXCISION HOMOTOPY DIMENSION

$$X = U \cup V$$

$$(U, U \cap V) \hookrightarrow (X, V)$$

induces iso

$$H_n(U, U \cap V) \xrightarrow{\cong} H_n(X, V)$$

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TRIPLE

$$X \supseteq Y \supseteq Z$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X,Z) \arrow[r] & H_n(X,Y) \arrow[r, "\Delta"] \arrow[d] & H_{n-1}(Y,Z) \arrow[d] \\
 & & H_{n-1}(Y) \arrow[ur] & H_{n-1}(X,Z) \arrow[d] \\
 & & & \vdots
\end{tikzcd}$$

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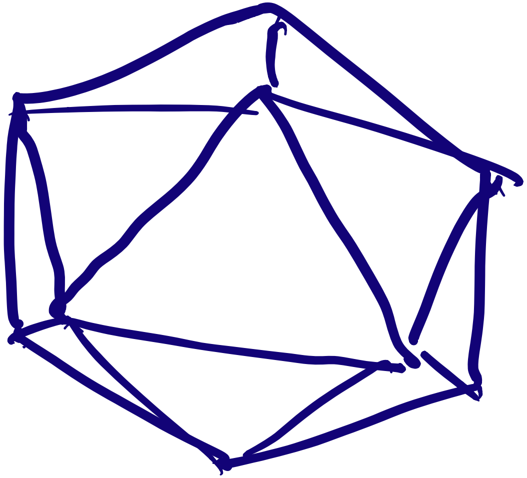
ICOSAHEDRON



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CELLULAR HOMOLOGY

Let X be a CW-cx.

$$X^{-1} \subseteq X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$$

$$\overset{\shortparallel}{\emptyset}$$

Cellular chain cx:

$$C_n(X) = H_n(X^n, X^{n-1})$$

$$C_n(X) \longrightarrow C_{n-1}(X)$$

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INFINITE COMPLEX PROJECTIVE SPACE

$$\begin{array}{ccc} \Delta_n \times S^{n-1} & \longrightarrow & X^{n-1} \\ \downarrow & & \downarrow \\ \Delta_n \times B^n & \longrightarrow & X^n \end{array}$$

$$C_n(X) \cong \mathbb{Z}\Delta_n$$

$$H_n(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & n \text{ even} \geq 0 \\ 0 & n \text{ odd, or } n < \infty \end{cases}$$

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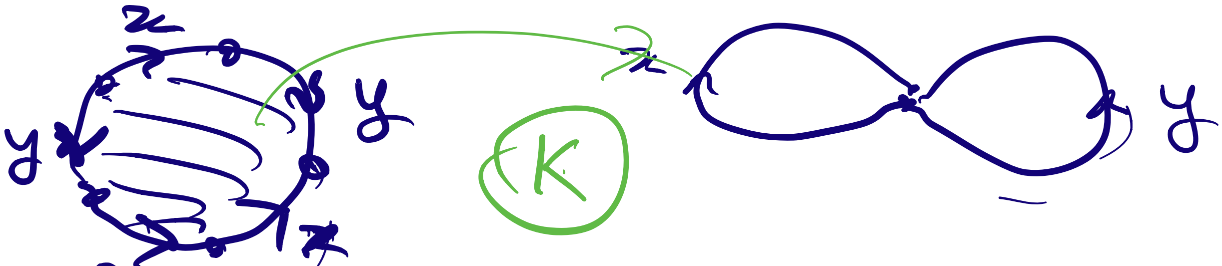
Eilenberg–Mac Lane spaces

TOP_*

$\pi_n(X, *)$ are abelian groups for $n \geq 2$

$\pi_1(X, *)$ can be any group

$\langle x, y \mid y^{-1} x y = x^2 \rangle$



$$\pi_n(X,*) \to H_n(X)$$

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van Kampen and Hurewicz

$$\pi_1(K) = \langle x, y \mid y^{-1}xy = x^2 \rangle$$

$$\underline{\text{disc}} \to \tilde{K} \longrightarrow K$$

So s'pose Δ_n are the n-cells of $\tilde{K}$

$$\cdots \to \mathbb{Z}\Delta_n \to \cdots\cdots \to \mathbb{Z}\Delta_2 \to \mathbb{Z}\Delta_1 \to \underset{\mathbb{Z}G}{\overset{\|}{\mathbb{Z}\Delta_0}} \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

{exact}

projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

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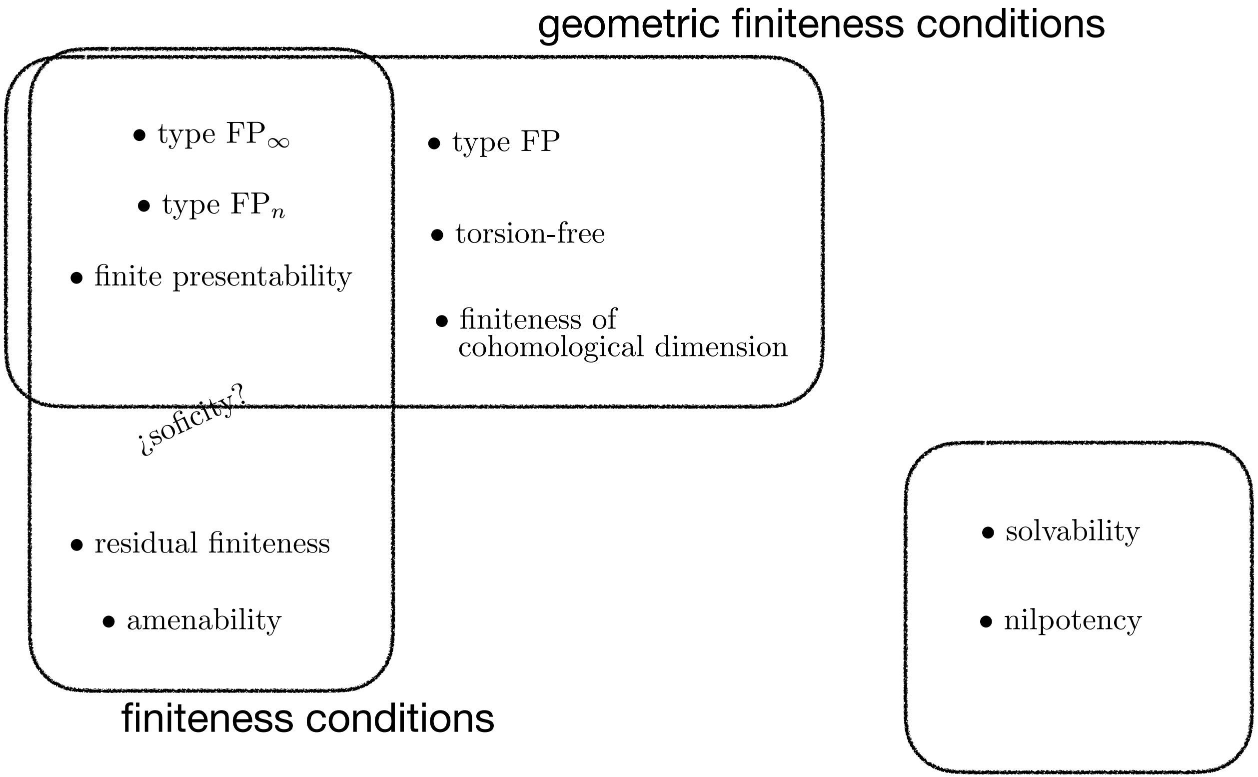
Geometric Finiteness Conditions

A finiteness condition is any property of gps that is true of all finite groups.

A gp is geometrically finite if it admits a finite EM-space

A geom'l finiteness conditi


geometric finiteness conditions
• type FP_∞
• type FP_n
• finite presentability
• type FP
• torsion-free
• finiteness of cohomological dimension
¿soficity?
• residual finiteness
• amenability
finiteness conditions
• solvability
• nilpotency

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Polyhedral products