The loop space of the group $SU(2)$ and the partially framed manifolds

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Let M^n be a smooth n-dimensional closed manifold.

The manifold M^n has a framing if there exists an isomorphism of vector bundles

$$(\mathcal{T} M^n) \oplus (N - n) \xrightarrow{\sim} (N),$$

where (k) is a trivial bundle.

The manifold M^n with a fixed framing is called a *fr*-manifold.

Let fr_n denotes the bordism group of framed manifolds.

The manifold M^n has a stably complex structure if there exist an N-dimensional complex vector bundle $\eta \to M^n$ and an isomorphism of real bundles

$$(\mathcal{T}M^n) \oplus (2N - n) \xrightarrow{\sim} r\eta,$$

where $r\eta$ is a bundle η regarded as real.

The manifold M^n with a fixed stably complex structure is called an U-manifold.

Let U_n denotes the bordism group of stably complex manifolds M^n.

Sp-manifolds and the group of quaternionic bordisms Sp_n are defined similarly.

A manifold M^n has a partially (with defect 1) framed U-structure if there exists an one-dimensional complex vector bundle $\xi \to M^n$ and the isomorphism

$$(\mathcal{T}M^n) \oplus \xi \oplus (2N - n - 2) \xrightarrow{\sim} (2N).$$

We denote such manifold by $(fr(1), U)$-manifold.

Let $\widetilde{\xi} \to M^n$ be a complex vector bundle such that $\xi \oplus \widetilde{\xi}$ is the trivial bundle.

We fix in M^n the structure of an U-manifold defined by the bundle $\widetilde{\xi}$, and will consider M^n as an U-manifold.

Then the one-dimensional complex bundle $\det \mathcal{T}M^n$ and the isomorphism $\xi^* \to \det \mathcal{T}M^n$ are defined, where ξ^* is a bundle, complex conjugate with ξ.

We introduce the bordism group $(fr(1), U)_n$ and the canonical homomorphism $(fr(1), U)_n \to U_n$.

If M^n has a SU-structure, then the $(fr(1), U)$-manifold is a framed.

Let M^{n+2} be some fr-manifold and $\xi \to M^{n+2}$ be some one-dimensional complex vector bundle.

Then the manifold $M^n \subset M^{n+2}$, dual to the bundle ξ, has the canonical structure of a $(fr(1), U)$-manifold.

Example.

Let T^2 be a smooth elliptic curve and $\eta \to T^2$ be an one-dimensional complex bundle such that $c_1(\eta)$ is a generator of the group $H^2(T^2; \mathbb{Z})$.
Then the $(fr(1), U)$-manifold $M^2 \subset M^4 = T^2 \times T^2$, dual to the bundle $\xi = \eta_1 \otimes \eta_2$, is a curve of genus 2.

Consider the one-dimensional quaternionic space $\mathbb{H}^1$ and set $S^3 = \{q \in \mathbb{H}^1 : |q| = 1\}$.

Let S^1 be a maximal commutative subgroup of $S^3 = Sp(1)$. Denote by $\mathbb{C}P^1_*$ a homogeneous space $Sp(1)/S^1$.

The tangent bundle $\mathcal{T}\mathbb{C}P^1_*$ is stably equivalent to the bundle $\eta \oplus \overline{\eta}$, where $\eta \to \mathbb{C}P^1_*$ is a complex line bundle such that $c_1(\eta)$ is a generator of $H^2(\mathbb{C}P^1_*; \mathbb{Z}) \simeq \mathbb{Z}$. We have $c_1(\mathcal{T}\mathbb{C}P^1_*) = 0$.

Let us introduce the submanifold $\mathcal{M}^{2n} \subset (\mathbb{C}P^1_*)^{n+1}$, that is dual to the complex line bundle $\eta_1 \otimes \cdots \otimes \eta_{n+1} \to (\mathbb{C}P^1_*)^{n+1}$.

Denote by $(fr(1), U)^{\triangleright}_{2n}$ the canonical image of $(fr(1), U)_{2n}$ in U_{2n}.

Theorem.

1. There are isomorphisms $(fr(1), U)^{\triangleright}_{2n} \simeq \mathbb{Z}$, $n \in N$.
2. The manifold $\mathcal{M}^{2n} \subset (\mathbb{C}P^1_*)^{n+1}$ is a representative of the generator of the group $(fr(1), U)^{\triangleright}_{2n}$.

Two-parameter complex Todd genus $U_{2n} \to \mathbb{Z}$ is defined by the series

$$Q(t) = \frac{t}{\beta(t;a,b)}, \quad \text{where} \quad \beta(t;a,b) = \frac{e^{at} - e^{bt}}{ae^{at} - be^{bt}}.$$

Important special cases:

1. $\beta = \beta(t;1,0) = 1 - e^{-t}$.
   $L_U(\beta)[M^{2n}] = Td(M^{2n})$ is the classical Todd genus of M^{2n}.
2. $\beta = \beta(t;1,-1) = \tanh(t)$.
   $L_U(\beta)[M^{2n}] = \tau(M^{2n})$ is the signature of M^{2n}.
3. $\beta = \beta(t;1,1) = \frac{t}{1+t}$.
   $L_U(\beta)[M^{2n}] = (-1)^n c_n(M^{2n})$ is the top Chern number of M^{2n}.

For complex (and almost complex) manifolds, the number $c_n(M^{2n})$ is equal to the Euler characteristic of M^{2n}.

Theorem.

1. $Td(\mathcal{M}^{2n}) = (-1)^n$.
2. $\tau(\mathcal{M}^{2n}) = \frac{4^{n+1}(4^{n+1}-1)}{2n+2} B_{2n+2}$,
   where B_k are the Bernoulli numbers.
3. $c_n(\mathcal{M}^{2n}) = (-1)^n(n+1)!$
4. If $[M^{4n}] \in (fr(1), U)_{4n}$ then $\tau(M^{4n})$ is divided by $\tau(\mathcal{M}^{2n})$.

Examples:

$Td(\mathcal{M}^2) = -1,\ c_1(\mathcal{M}^2) = -2$,
hence, $\mathcal{M}^2$ is an oriented surface of genus 2; $[\mathcal{M}^2] = -[\mathbb{C}P^1]$.

$Td(\mathcal{M}^4) = 1,\ \tau(\mathcal{M}^4) = -2,\ c_2(\mathcal{M}^4) = -6$,
hence, $[\mathcal{M}^4] = 3[\mathbb{C}P^1]^2 - 2[\mathbb{C}P^2]$.

$\tau(\mathcal{M}^8) = 16,\ \tau(\mathcal{M}^{12}) = -16 \cdot 17$.

A manifold M^n has a stable partially (with defect 1) framed Sp-structure if there exists an one-dimensional quaternionic bundle $\zeta \to M^n$ and the isomorphism

$$(\mathcal{T}M^n) \oplus \zeta \oplus (4N - n - 4) \xrightarrow{\sim} (4N).$$

We denote such manifold by $(fr(1), Sp)$-manifold.

Let $\widetilde{\zeta} \to M^n$ be a quaternionic bundle such that $\zeta \oplus \widetilde{\zeta}$ is the trivial bundle. We fix in M^n the structure of an Sp-manifold defined by the bundle $\widetilde{\zeta}$, and will consider M^n as an Sp-manifold.

We introduce the bordism group $(fr(1), Sp)_n$ and the canonical homomorphism $(fr(1), Sp)_n \to Sp_n$.

Let M^{n+4} be some fr-manifold and $\zeta \to M^{n+4}$ be some one-dimensional quaternionic bundle. Then the manifold $M^n \subset M^{n+4}$, dual to the bundle ζ, has the canonical structure of a $(fr(1), Sp)$-manifold.

For any n the manifold $(\mathbb{C}P^1_*)^n$ is a Sp-manifold.
Let us introduce the Sp-submanifold $\mathcal{N}^{2n} \subset (\mathbb{C}P^1_*)^{n+2}$, $n \geqslant 0$, that is dual to the quaternionic bundle $\eta \oplus \overline{\eta} \to (\mathbb{C}P^1_*)^{n+2}$, where $\eta = \eta_1 \otimes \cdots \otimes \eta_{n+2}$. We have $\mathcal{N}^0 = 2$.

Let us denote by $(fr(1), Sp)^{\triangleright}_{4n}$ canonical image of $(fr(1), Sp)_{4n}$ in the group U_{4n}.

Theorem.

1. There are isomorphisms $(fr(1), Sp)^{\triangleright}_{4n} \simeq \mathbb{Z}$ $n \in N$.
2. The manifold $\mathcal{N}^{4n} \subset (\mathbb{C}P^1_*)^{2n+2}$ is a representative of the generator of the group $(fr(1), Sp)^{\triangleright}_{4n}$.
3. $Td(\mathcal{N}^{4n}) = (-1)^n \varepsilon_n$, where $\varepsilon_{2k-1} = 2$ and $\varepsilon_{2k} = 1$, $k = 1, 2, \ldots$.
4. $\tau(\mathcal{N}^{4n}) = (-1)^{n+1} \frac{4^{n+2}(4^{n+2}-1)}{2(n+2)} B_{2(n+2)}$, where B_{2k} are the Bernoulli numbers.
5. $c_{2n}(\mathcal{N}^{4n}) = (-1)^n (2n+2)!$

Examples:

(1) $Td(\mathcal{N}^4) = -2$; $\tau(\mathcal{N}^4) = 16$; $c_2(\mathcal{N}^4) = -4!$

(2) $Td(\mathcal{N}^8) = 1$; $\tau(\mathcal{N}^8) = 16 \cdot 17$; $c_4(\mathcal{N}^8) = 6!$

(3) $Td(\mathcal{N}^{12}) = -2$; $\tau(\mathcal{N}^{12}) = 16^2 \cdot 31$; $c_6(\mathcal{N}^{12}) = -8!$

The Todd genus of any $(8m+4)$-dimensional Sp-manifold is even.

Corollary.

The manifold $\mathcal{N}^4$ is the representative of generator of the group $Sp_4 \simeq \mathbb{Z}$.

Homotopy interpretation of bordism groups

$$fr_n = \lim_{N\to\infty} [S^{N+n}, S^N]_*$$

$$U_n = \lim_{N\to\infty} [S^{2N+n}, MU(N)]_*$$

$$Sp_n = \lim_{N\to\infty} [S^{4N+n}, MSp(N)]_*$$

$$(fr(1), U)_n = \lim_{N\to\infty} [S^{2N+n}, \Sigma^{2N-2}\mathbb{C}P^\infty]_*$$

$$(fr(1), Sp)_n = \lim_{N\to\infty} [S^{4N+n}, \Sigma^{4N-4}\mathbb{H}P^\infty]_*$$

[1] L. S. Pontryagin, *Smooth manifolds and their applications in homotopy theory.*, Trudy Mat. Inst. Steklov., 45, Acad. Sci. USSR, Moscow, 1955, 3–139.

[2] R. Thom, *Quelques propriétes globales des variétés différentiables.*, Commun. Math. Helv., 28 (1954), 17–86.

The loop spaces ΩG of compact Lie groups G are classical objects of algebraic topology.

The Hopf algebra theory and Morse theory have found fundamental applications in problems on cohomology rings $H^*(\Omega G; \mathbb{Z})$ and Pontryagin algebras $H_*(\Omega G; \mathbb{Z})$ (see [3]).

Analytical and differential-geometrical properties of the loop spaces ΩG attracted the attention of specialists in the theory of integrable systems, string theory, and the theory of infinite-dimensional Kähler's manifolds (see [4], [5]).

[3] R. Bott, *The space of loops on a Lie group.*, Michigan Math. J., 5 (1958), 35–61.

[4] M. F. Atiyah, A. N. Pressley *Convexity and loop groups.*, in Arithmetic and Geometry, Springer, 1983, 33–63.

[5] A. Pressley, G. Segal, *Loop groups.*, Oxford University Press (1988).

Theorem (R. Bott, 1958).

1. The Pontryagin algebra $H_*(\Omega SU(n+1);\mathbb{Z})$, $n \geqslant 1$, is isomorphic to the Hopf algebra on the graded ring $\mathbb{Z}[b_1, \ldots, b_n]$, $\deg b_k = 2k$, with the diagonal

$$\Delta b_k = \sum_{i+j=k} b_i \otimes b_j .$$

2. There is an embedding

$$j_n\colon \mathbb{C}P^n \to \Omega SU(n+1)$$

such that $b_k = (j_n)_* a_k$, $k = 1, \ldots, n$, where a_k is a generator of group $H_{2k}(\mathbb{C}P^n;\mathbb{Z})$.

Let us consider the universal complex n-dimensional bundle $\eta(n) \to BU(n)$. We have

$$H_*(BU(n);\mathbb{Z}) \simeq \mathbb{Z}[c_1,\dots,c_n],$$

where $c_k = c_k(\eta(n)) \in H^{2k}(BU(n);\mathbb{Z})$ are the Chern classes of $\eta(n)$.

In the case $n > 1$, the space $BU(n)$ is not a H-space, but the ring $H^*(BU(n);\mathbb{Z})$ has the structure of a Hopf algebra with diagonal

$$\Delta c_k = \sum_{i+j=k} c_i \otimes c_j.$$

Therefore, the Pontryagin-Hopf algebra $H_*(\Omega SU(n+1);\mathbb{Z})$ is isomorphic to the Hopf algebra $H^*(BU(n);\mathbb{Z})$.

From the point of view of the prodlems of our talk, the loop space on the 3-dimensional sphere S^3 is remarkable object.
The manifold S^3, as a Lie group, is $SU(2) = SP(1)$,
and on the other hand, as a CW-complex, it is ΣS^2.

We use algebraic-topological and geometric constructions to describe the cobordism ring $U^*(\Omega S^3)$ with the action of the Landweber–Novikov algebra S and the bordism group $U_*(\Omega S^3)$ as the ring with the Pontryagin multiplication.

We describe the canonical map $\varphi\colon \Omega S^3 \to \mathbb{C}P^\infty$ and use its to calculate the cobordism classes of U-manifolds $\mathcal{M}^{2n}$ and Sp-manifolds $\mathcal{N}^{4n}$.

James spaces.

Let X be some CW-complex with a base point $*$. The James space $J(X)$ is the free noncommutative monoid generated by the space X, in which point $*$ plays the role of unit. This space is a CW-complex $J(X) = \lim_{n\to\infty} J_n(X)$. Here $J_n(X)$ is the image of the projection $j_n\colon X^n \to J_n(X)\ :\ j_n\colon (x_1, \dots, x_n) \to (x_1\sharp\cdots\sharp x_n)$, where $\sharp$ is the symbol of multiplication in the monoid $J(X)$.

The direct limit is taken with respect to the embeddings $i_n\colon J_n(X) \subset J_{n+1}(X)$ that are induced by the embeddings $X^n \subset X^{n+1}\colon (x_1, \dots, x_n) \to (x_1, \dots, x_n, *)$.

Theorem (I. M. James, 1955).

1. There is a homotopy equivalence $j_X\colon J(X) \to \Omega\Sigma X$;
2. There is a homotopy equivalence $\Sigma j_X\colon \Sigma J(X) \to \sum\limits_{n\geqslant 1} \vee X^{(n)}$,

   where $X^{(n)} = X \wedge \cdots \wedge X$ is the n-th smash power.

Dold–Thom spaces.

The Dold–Thom space $DT(X)$ is the free commutative monoid generated by the space X, in which the base point $*$ plays the role of unit. This space is a CW-complex $DT(X) = \lim_{n\to\infty} DT_n(X)$. Here $DT_n(X) = J_n(X)/S_n$.

The Dold–Thom space $DT(X)$ is also called the infinite symmetric product and is denoted by $SP(X)$.

The finite CW-complexes $J_n(X)$ and $DT_n(X)$ together with the projection $J_n(X) \to DT_n(X)$ define two functors $J(X)$ and $DT(X)$ with values in the category of monoids together with the "abelinization" morphism $J(X) \to DT(X)$.

Set $A_n(X) = H_n(X;\mathbb{Z})$ and $A(X) = \sum\limits_{n\geqslant 0} A_n(X) = H_*(X;\mathbb{Z})$.

Let us introduce the Eilenberg–MacLane space

$K(A(X)) = \lim\limits_{n\to\infty} \prod\limits_{m=1}^{n} K(A_m(X), m)$.

If $X = S^n$, then $K(A(S^n)) = K(\mathbb{Z}; n)$.

Theorem (A. Dold, R. Thom, 1958).

There is the functorial homotopy equivalence
$k(X)\colon DT(X) \to K(A(X))$.

Since $\mathrm{Sym}^{\mathrm{n}}(\mathbb{C}\mathrm{P}^1) = \mathbb{C}\mathrm{P}^{\mathrm{n}}$ then $\mathbb{C}P^\infty = \lim\limits_{n\to\infty} \mathbb{C}P^n$ can be identified with $DT(\mathbb{C}P^1)$.

We identify the space $X = \Omega S^3$ with the free monoid $J(S^2)$ and the space $\mathbb{C}P^\infty$ with the commutative free monoid $DT(S^2)$.

Let $\varphi\colon \Omega S^3 \to \mathbb{C}P^\infty$ denote the map defined by the abelinization homomorphism $J(S^2) \to DT(S^2)$.

The canonical embedding $\mathbb{C}P^1_* \subset \Omega S^3$ defines a generator $w \in \widetilde{U}_2(\Omega S^3) \simeq \mathbb{Z}$, that under $\varphi_*\colon U_*(\Omega S^3) \to U_*(\mathbb{C}P^\infty)$ pass in a generator $v \in \widetilde{U}_2(\mathbb{C}P^\infty) \simeq \mathbb{Z}$.

In the homology of $J(S^2) = \lim\limits_{n\to\infty} J_n(S^2)$, the generator of the group $H_{2n}(J(S^2))$ is realized by the image of the fundamental cycle of the manifold $(\mathbb{C}P^1_*)^n$ under $j_n\colon (\mathbb{C}P^1_*)^n \to J_n(S^2)$.

Theorem.

With respect to the multiplication in bordisms induced by the structure of monoid $J(S^2) \simeq \Omega S^3$, there exists an isomorphism $U_*(\Omega S^3) \simeq \Omega_U[w]$.

Thus, $U_*(\Omega S^3)$ is a free Ω_U-module with generators $w_{2n} \in U_{2n}(\Omega S^3)$, $n = 0, 1, \ldots$, where $w_{2n} = w^n$.

The standard embeddings $S^{2n} \subset MU(n)$ define the generators $u(n)$ of the groups $U^{2n}(S^{2n}) \simeq \mathbb{Z}$.

Using the homotopy equivalence $\Sigma j_{S^2} \colon \Sigma J(S^2) \to \Sigma\big(\bigvee_{n\geqslant 1} S^{2n}\big)$, we take the classes $b_{2n} \in U^{2n}(\Omega S^3)$ such that

$$(\Sigma j_{S^2})^* \sigma u(n) = \sigma b_{2n},$$

as generators of the free Ω_U-module $U^*(\Omega S^3)$, where σ is the suspension isomorphism.

Let us describe explicitly the cap-product

$$\cap\colon U_n(X) \otimes U^m(X) \to U_{n-m}(X).$$

Let M^n be some closed U-manifold with a fixed U-orientation $\alpha(\nu)$ of a normal bundle. Let us denote by $D = D(\alpha(\nu))$ the Poincare duality isomorphism $U^k(M^n) \to U_{n-k}(M^n)$.

Set $a \in U_n(X)$ and $b \in U^m(X)$. Let us choose a representative $f\colon M^n \to X$ of the bordism class a and put, by definition,

$$a \cap b = f_* D f^* b \in U_{n-m}(X).$$

The fundamental class $\langle M^n \rangle \in U_n(M^n)$ is defined by the identity map $M^n \to M^n$. We obtain $\langle M^n \rangle \cap b = Db$ for any $b \in U^m(M^n)$.

Consider the homomorphism $\varepsilon\colon U_*(X) \to \Omega_U$ induced by the map $X \to \mathrm{pt}$. We obtain the Ω_U-bilinear scalar product $(\cdot,\cdot)\colon U_n(X) \otimes U^m(X) \to \Omega^U_{n-m}$ by the formula $(a, b) = \varepsilon(a \cap b)$.

Let us consider the Ω_U-bilinear scalar product

$$(\cdot,\cdot)\colon U_n(\Omega S^3)\otimes U^m(\Omega S^3)\to \Omega^U_{n-m}.$$

Let $\{w_{2n}\}$ and $\{b_{2n}\}$ be the bases of the free Ω_U-modules $U_*(\Omega S^3)$ and $U^*(\Omega S^3)$. By construction, we have $(w_{2n}, b_{2m}) = \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker symbol.

Let $u \in U^2(\mathbb{C}P^\infty)$ be the first Chern-Conner-Floyd class of the universal one-dimensional bundle $\eta \to \mathbb{C}P^\infty$. Then by definition of manifolds $\mathcal{M}^{2n}$, there is the formula

$$(w_{2n+2}, \varphi^* u) = [\mathcal{M}^{2n}],\ n \geqslant 1.$$

Theorem.

$$\varphi_U^* u = b_2 + \sum_{n\geqslant 1}[\mathcal{M}^{2n}]b_{2n+2}.$$

The spectra $MU = \{MU(n)\}$ and $K(\mathbb{Z}) = \{K(\mathbb{Z}; 2n)\}$ define theories $U^*(\cdot)$, $U_*(\cdot)$ and $H^*(\cdot; \mathbb{Z})$, $H_*(\cdot; \mathbb{Z})$, respectively.

The cohomology classes $u^H_{2n} \in H^{2n}(MU(n); \mathbb{Z})$ define

$$\mu^H_U \colon U^*(\cdot) \to H^*(\cdot; \mathbb{Z}) \text{ and } \mu^H_U \colon U_*(\cdot) \to H_*(\cdot; \mathbb{Z}).$$

Let $\mathcal{A}_U = \Omega_U S$ be the algebra of all cohomological operations in the theory $U^*(\cdot)$ and S be the Landweber–Novikov algebra.

Theorem.

1. There is an isomorphism $U^*(\Omega S^3) \simeq \Omega_U[[b_2, \ldots, b_{2n}, \ldots]]/J$, where $J = \{b_2^n - n! b_{2n}\}$, $n = 2, 3, \ldots$; and therefore $b_{2i} b_{2j} = \binom{i+j}{i} b_{2(i+j)}$.
2. The homomorphism $\mu^H_U \colon U^*(\Omega S^3) \to H^*(\Omega S^3; \mathbb{Z})$ defines the isomorphism of $\mathcal{A}_U$-modules $U^*(\Omega S^3) \to H^*(\Omega S^3; \Omega_U) = \Omega_U[[\widehat{b}_2, \ldots, \widehat{b}_{2n}, \ldots]]/\widehat{J}$, where $\widehat{b}_{2n} = \mu^H_U b_{2n}$.

The standard embeddings $Sp(n) \subset U(2n)$ define mappings of the classifying spaces $f_n \colon BSp(n) \to BU(2n)$.

Let $\eta(2n) \to BU(2n)$ and $\zeta(n) \to BSp(n)$ be universal bundles. We have $f_n^*\eta(2n) = \zeta(n)$. The mapping of the Thom spaces $Tf_n \colon MSp(n) \to MU(2n)$ is defined. The Thom spectrum $MSp = \{MSp(n)\}$ defines theories $Sp^*(\cdot)$ and $Sp_*(\cdot)$.

The maps Tf_n define maps of the spectra $Tf \colon MSp \to MU$ and the transformations

$$\mu_{Sp}^U \colon Sp^*(\cdot) \to U^*(\cdot) \text{ and } \mu_{Sp}^U \colon Sp_*(\cdot) \to U_*(\cdot).$$

Let $t(\eta(2n)) \in U^{4n}(MU(2n))$ and $t(\zeta(n)) \in Sp^{4n}(MSp(n))$ be the Thom classes of universal bundles.

There is the formula $\mu_{Sp}^U t(\zeta(n)) = (Tf_n)^* t(\eta(2n))$.

The Chern-Conner-Floyd classes $c_k^U(\eta)$ of complex bundles $\eta \to X$ are defined in the theory $U^*(\cdot)$.
The Borel-Conner-Floyd classes $p_k^{Sp}(\zeta)$ of quaternionic bundles $\zeta \to X$ are defined in the theory $Sp^*(\cdot)$.

There exists the formula $\mu_{Sp}^U p_k^{Sp}(\zeta(n)) = f_n^* c_{2k}^U(\eta(2n))$.

We have
$$\mathbb{C}P^\infty = BU(1) \simeq MU(1) \text{ and } \mathbb{H}P^\infty = BSp(1) \simeq MSp(1).$$

The algebra $\mathcal{A}_{Sp}$ of all cohomological operations in theory $Sp^*(\cdot)$ has the form $\Omega_{Sp} S^{Sp}$, where the algebra of operations S^{Sp} is constructed using the classes $p_k^{Sp}(\zeta)$ in the same way as the Landweber-Novikov algebra $S = S^U$ is constructed using the classes $c_k^U(\eta)$.

The embedding $S^1 \subset Sp(1)$ defines the bundle $\pi\colon \mathbb{C}P^\infty \to \mathbb{H}P^\infty$ with the fiber $S^3/S^1 \simeq \mathbb{C}P^1_*$. We have $\pi^*\zeta(1) = \eta \oplus \overline{\eta}$.
Let us denote by $\psi\colon \Omega S^3 \to \mathbb{H}P^\infty$ the composition $\pi\varphi$.
According to p.2 of I.M.James' Theorem (Slide 16) for $X = S^2$ the ring $Sp^*(\Omega S^3)$ is a free Ω_{Sp}-module.

The ring Ω_{Sp} has 2-torsion; therefore, the ring $Sp^*(\mathbb{C}P^\infty)$ is not a free Ω_{Sp}-module.

We have $Sp^*(\mathbb{H}P^\infty) = \Omega_{Sp}[[x]]$, where $x = p_1^{Sp}(\zeta(1))$, $\mu^U_{Sp}x = u\bar{u}$, and $u = c_1^U(\eta(1)) \in U^2(\mathbb{C}P^\infty)$.

Let us consider the Ω_{Sp}-bilinear scalar product

$$(\cdot,\cdot)\colon Sp_n(\Omega S^3) \otimes Sp^m(\Omega S^3) \to \Omega^{Sp}_{n-m}.$$

Let $\{w_{2n} \in Sp_{2n}(\Omega S^3)\}$ and $\{b_{2n} \in Sp^{2n}(\Omega S^3)\}$ are dual bases of free Ω_{Sp}-modules.
By definition of manifolds $\mathcal{N}^{2n} \subset (\mathbb{C}P^1_*)^{n+2}$, there is the formula

$$(w_{2n+4}, \psi^*_{Sp}x) = [\mathcal{N}^{2n}], \ n = 0, 1, \ldots$$

Theorem.

$$\psi^*_{Sp} x = 2b_4 + \sum_{n \geqslant 1} [\mathcal{N}^{2n}] b_{2n+4}.$$

Let us identify the ring $\operatorname{Im} \mu^{\mathrm{U}}_{\mathrm{Sp}} \Omega_{\mathrm{Sp}}$ with the ring $\widehat{\Omega}_{Sp} = \Omega_{Sp}/\, Tors$.

Theorem.

1. There is an isomorphism $Sp^*(\Omega S^3) \simeq \Omega_{Sp}[[b_2, \ldots, b_{2n}, \ldots]]/J$, where $J = \{b_2^n - n! b_{2n}\}$, $n = 2, 3, \ldots$; and therefore $b_{2i} b_{2j} = \binom{i+j}{i} b_{2(i+j)}$.
2. The homomorphism $\mu^H_{Sp} \colon Sp^*(\Omega S^3) \to H^*(\Omega S^3; \mathbb{Z})$ defines the isomorphism of $\mathcal{A}_{Sp}$-modules $Sp^*(\Omega S^3) \to H^*(\Omega S^3; \Omega_{Sp}) = \Omega_{Sp}[[\widehat{b}_2, \ldots, \widehat{b}_{2n}, \ldots]]/\widehat{J}$, where $\widehat{b}_{2n} = \mu^H_{Sp}\, b_{2n}$.

The Chern–Dold character in cobordism theory (see [6]) is the multiplicative transformation of cohomology theories

$$\mathrm{ch_U}\colon \mathrm{U}^*(\cdot) \to \mathrm{H}^*(\cdot; \Omega_\mathrm{U} \otimes \mathbb{Q})$$

that is uniquely determined by the condition

$$\mathrm{ch_U}[\mathrm{M}^{2n}] = [\mathrm{M}^{2n}] \quad \text{for any } [M^{2n}] \in \Omega_U^{-2n}.$$

The algebra of all cohomological operations in $H^*(\cdot; \Omega_U \otimes \mathbb{Q})$ coincides with the algedra $\mathcal{A}_U \otimes \mathbb{Q}$, whose action is determined by the action of $\mathcal{A}_U$ on Ω_U.

Properties:

1. The transformation $\mathrm{ch_U}$ commutes with the action of $\mathcal{A}_U$.
2. Let X be some finite CW-complex. Then

$$\mathrm{ch_U} \otimes \mathbb{Q}\colon \mathrm{U}^*(\mathrm{X}) \otimes \mathbb{Q} \to \mathrm{H}^*(\mathrm{X}; \Omega_\mathrm{U} \otimes \mathbb{Q})$$

is an isomorphism of $\mathcal{A}_U$-modules.

[6] V. M. Buchstaber, *The Chern-Dold Character in Cobordisms, I*, Math.USSR Sbornik, 12:4 (1970), 573–594.

3. Let X be some CW-complex such that $H^*(X;\mathbb{Z})$ is torsion-free. Then $\mathrm{ch_U}\otimes\mathbb{Q}\colon \mathrm{U^*(X)}\otimes\mathbb{Q}\to \mathrm{H^*(X;\Omega_U}\otimes\mathbb{Q})$ is an isomorphism of $\mathcal{A}_U$-modules.

4. Set $u=c_1^U(\eta(1))\in U^2(\mathbb{C}P^\infty)$. The transformation $\mathrm{ch_U}$ is uniquely determined by series $\mathrm{ch_U u = f(t) = t + \sum\limits_{n\geqslant 1} a_{2n}t^{n+1}}$,

where $a_{2n}=\frac{[M_*^{2n}]}{(n+1)!}$, $[M_*^{2n}]\in\Omega_U^{-2n}$.

5. For any $[M^{2n}]\in\Omega_U^{-2n}$, there is the formula

$$\mathrm{ch_U[M^{2n}] = [M^{2n}] = \sum_{|\omega|=n} c_\omega(\nu(M^{2n}))\, a_\omega},$$

where $\omega=(i_1,\ldots,i_k)$, $i_1\geqslant\ldots\geqslant i_k>0$, $|\omega|=\sum i_k=n$, $a_\omega=a_{i_1}\cdots a_{i_k}$, and $c_\omega(\nu(M^{2n}))$ are the Chern numbers of the normal bundle of a stably complex manifold M^{2n}.

6. From the formula $[M_*^{2n}] = \sum\limits_{|\omega|=n} c_\omega(\nu(M_*^{2n}))\, a_\omega$ follows:

$$\Omega_U \otimes \mathbb{Q} = \mathbb{Q}\big[[M_*^2], \dots, [M_*^{2n}], \dots\big],$$

$$c_\omega(\nu(M_*^{2n})) = 0,\ \omega \neq (n), \quad c_{(n)}(\nu(M_*^{2n})) = (n+1)!$$

In the Landweber-Novikov algebra $S = \sum_{n \geqslant 0} S_n$, the groups S_n are additively generated by the operations s_ω, $|\omega| = n$, which act on the classes $[M^{2n}] \in \Omega_U^{-2n}$ by the formula

$$s_\omega[M^{2n}] = c_\omega(\nu(M^{2n})).$$

7. The Chern–Dold character coefficients $[M_*^{2n}] \in \Omega_U$ are uniquely determined by the conditions

$$s_\omega[M_*^{2n}] = 0,\ \omega \neq (n), \quad s_{(n)}[M_*^{2n}] = (n+1)!$$

Set $\Omega_U(\mathbb{Z}) = \mathbb{Z}\left[\frac{[M_*^2]}{2}, \dots, \frac{[M_*^{2n}]}{(n+1)!}, \dots\right] \subset \Omega_U \otimes \mathbb{Q}$.

8. A multiplicative transformation of cohomology theories

$$\widehat{\mathrm{ch}}_U \colon U^*(\cdot) \to H^*(\cdot; \Omega_U(\mathbb{Z}))$$

is defined such that ch_{U} decomposes into a composition

$$\mathrm{ch}_U \colon U^*(X) \to H^*(X; \Omega_U(\mathbb{Z})) \to H^*(X; \Omega_U \otimes \mathbb{Q}).$$

9. Let $H^*(X;\mathbb{Z})$ be torsion-free. Then the transformation $\widehat{\mathrm{ch}}_U$ defines the isomorphism

$$\widehat{\mathrm{ch}}_U \otimes 1 \colon U^*(X) \otimes_{\Omega_U} \Omega_U(\mathbb{Z}) \to H^*(X; \Omega_U(\mathbb{Z})).$$

Any Hirzebruch genus L defines a ring homomorphism $L\colon \Omega_U \to \mathbb{Q}$ and is defined by the series $Q(t) = 1 + \sum\limits_{k\geqslant 1} \alpha_{2k} t^k \in \mathbb{Q}[[t]]$.

Put $\frac{1}{Q(t)} = 1 + \sum\limits_{n\geqslant 1} \beta_{2n} t^n$.

10. For the coefficients $[M_*^{2n}]$ of the Chern–Dold character ch_{U}, there is the formula

$$L[M_*^{2n}] = (n+1)!\beta_{2n}.$$

Example.

The Todd genus is defined by the series $Q(t) = \frac{t}{1-e^{-t}}$.
Therefore $L[M_*^{2n}] = (-1)^n$.

Let $g(u) = u + \sum [\mathbb{C}P^n] \frac{u^{n+1}}{n+1}$ be the logarithm of the formal group of geometrical cobordisms.

11. There is the formula $\widehat{\mathrm{ch}}_U g(u) = f(t) + \sum [\mathbb{C}P^n] \frac{f(t)^{n+1}}{n+1} = t$.

Corollary.

Series $f(t)$ is functionally inverse to the Mishchenko's series $g(u)$.

Under the $\mathcal{A}_U$-isomorphism

$$\widehat{\mathrm{ch}}_U \otimes 1 \colon U^*(\mathbb{C}P^\infty) \otimes_{\Omega_U} \Omega_U(\mathbb{Z}) \to H^*(\mathbb{C}P^\infty; \Omega_U(\mathbb{Z})),$$

the series $g(u)$ goes to the element $t \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

The Chern–Dold character

$$\mathrm{ch_{Sp}}\colon \mathrm{Sp}^*(\cdot) \to \mathrm{H}^*(\cdot;\Omega_{\mathrm{Sp}}\otimes\mathbb{Q})$$

in theory $Sp^*(\cdot)$ is given by series

$$\mathrm{ch_{Sp}x} = \mathrm{h(z)} = \mathrm{z} + \sum_{\mathrm{n}\geqslant 1} \mathrm{b_n z^{n+1}},$$

where $x = p_1^{Sp}(\zeta(1)) \in Sp^4(\mathbb{H}P^\infty)$ and $z = p_1^{H}(\zeta(1)) \in H^4(\mathbb{H}P^\infty)$. Here $b_n = \frac{[N_*^{4n}]}{(2n+2)!}$, where $[N_*^{4n}] \in \Omega_{Sp}^{-4n}$.

For any class $[N^{4n}] \in \Omega_{Sp}^{-4n}/\,Tors$, there is the formula

$$\mathrm{ch_{Sp}[N^{4n}]} = \mathrm{[N^{4n}]} + \sum_{|\omega|=\mathrm{n}} \mathrm{p}_\omega(\nu(\mathrm{M^{4n}}))\,\mathrm{b}_\omega,$$

where $p_\omega(\nu(M^{4n}))$ is the characteristic Borel numbers.

Let the Hirzebruch genus $L\colon \Omega_U \to \mathbb{Q}$ be given by the series $Q(t)$. Put $\frac{1}{Q(t)Q(-t)} = 1 + \sum\limits_{n\geqslant 1} \gamma_{4n} z^n$, where $z = -t^2$ and $t = i\sqrt{z}$.

12. For the coefficients $[N_*^{4n}]$ of the Chern–Dold character $\mathrm{ch}_{\mathrm{Sp}}$, there is the formula

$$L[N_*^{4n}] = (2n+2)!\,\gamma_{4n}.$$

Let $L = Td$ be the Todd genus. Then

$$\frac{1}{Q(t)Q(-t)} = \frac{2}{z}(1 - \cos\sqrt{z}).$$

Let $L = \tau$ be the signature. Then

$$\frac{1}{Q(t)Q(-t)} = \frac{1}{z}\Big(\frac{d}{d\sqrt{z}}\tan\sqrt{z} - 1\Big).$$

Let $L = c_n$ be the top Chern classes. Then

$$\frac{1}{Q(t)Q(-t)} = \frac{1}{1+z}.$$

The geometric realization of coefficients of the Chern–Dold character ch_{U}.

Theorem.

$(fr(1), U)$-manifold

$$\mathcal{M}^{2n} \subset (\mathbb{C}P^1_*)^{n+1}, \; n = 0, 1, \ldots,$$

is a representative of the U-cobordism class $[M^{2n}_*]$.

Let T^2 be a smooth elliptic curve and $\eta \to T^2$ be an one-dimensional complex bundle such that $c_1(\eta)$ is a generator of the group $H^2(T^2; \mathbb{Z})$.

Theorem.

The $(fr(1), U)$-manifold $M^{2n} \subset M^{2n+2} = (T^2)^{n+1}$, dual to the bundle $\xi = \xi_{n+1} = \eta_1 \otimes \cdots \otimes \eta_{n+1}$, is a representative of the U-cobordism class $[M^{2n}_*]$.

The geometric realization of coefficients of the Chern–Dold character $\mathrm{ch}_{\mathrm{Sp}}$.

Theorem.

$(fr(1), Sp)$-manifold

$$\mathcal{N}^{4n} \subset (\mathbb{C}P^1_*)^{2n+2},\ n = 0, 1, \ldots,$$

is a representative of the Sp-cobordism class $[N^{4n}_*]$.

Let T^2 be a smooth elliptic curve and $\eta \to T^2$ be an one-dimensional complex bundle such that $c_1(\eta)$ is a generator of the group $H^2(T^2; \mathbb{Z})$.

Theorem.

The $(fr(1), Sp)$-manifold $N^{4n} \subset M^{4n+4} = (T^2)^{2n+2}$, dual to the bundle $\zeta = \xi_{2n+2} \oplus \bar{\xi}_{2n+2}$, where $\xi_{2n+2} = \eta_1 \otimes \cdots \otimes \eta_{2n+2}$, is a representative of the Sp-cobordism class $[N^{4n}_*]$.

Thank you for the attention!