Ivan Arzhantsev (HSE University, Moscow)

Infinite transitivity, finite generation, and Demazure roots

with Karine Kuyumzhiyan and Mikhail Zaidenberg

Advances in Mathematics 351 (2019) 1–32

14.05.2020

Infinite Transitivity

Definition

Let G be a group, X a set, and m a positive integer. An action $G \times X \to X$ is called *m-transitive* if for any two tuples $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$ of pairwise distinct points on X there is an element $g \in G$ such that $(ga_1, \ldots, ga_m) = (b_1, \ldots, b_m)$.

Definition

An action $G \times X \to X$ is *infinitely transitive* if it is m-transitive for any positive integer m.

Example

1) Let X be an infinite set and G the group of all permutations on X.
2) Let X be an infinite set and G the group of all permutations with finite support on X.

Affine Spaces

Theorem

The group $\mathrm{Aut}(\mathbb{A}^n)$ *is infinitely transitive on* $\mathbb{A}^n$ *for any* $n \geqslant 2$.

Idea ($n = 2$): use parallel translations $(x_1 + a, x_2)$, $(x_1, x_2 + b)$ and their replicas $(x_1 + af_1(x_2), x_2)$, $(x_1, x_2 + bf_2(x_1))$, where $a, b \in K$.

Example

The group $\mathrm{Aut}(\mathbb{A}^1)$ is isomorphic to $K^\times \leftthreetimes K$. It is 2-transitive, but not 3-transitive on $\mathbb{A}^1$.

General Problems

Let X be an affine algebraic variety over the field $\mathbb{C}$.

When the group $\mathrm{Aut}(X)$ of polynomial automorphisms of X is infinitely transitive on X?

If X is singular, we ask this question for the smooth locus X^{reg} .

Idea: to use $\mathbb{G}_a$-subgroups in the group $\mathrm{Aut}(X)$ and their replicas. Here $\mathbb{G}_a = (\mathbb{C}, +)$.

Notation: $\mathrm{SAut}(X)$ is the subgroup of $\mathrm{Aut}(X)$ generated by all $\mathbb{G}_a$-subgroups.

Locally Nilpotent Derivations

Definition

A derivation $D\colon A \to A$ of an algebra A is *locally nilpotent* if for any $a \in A$ there is a positive integer k such that $D^k(a) = 0$.

Locally nilpotent derivations on $\mathbb{C}[X] \Leftrightarrow \mathbb{G}_a$-subgroups in $\mathrm{Aut}(X)$

$$D \in \mathrm{LND}(\mathbb{C}[X]) \iff \exp(\mathbb{C}D) \subseteq \mathrm{Aut}(X)$$

If $D \in \mathrm{LND}(A)$ and $f \in \mathrm{Ker}(D)$, then $fD \in \mathrm{LND}(A)$.

$\mathbb{G}_a$-subgroups corresponding to LNDs of the form fD are *replicas* of the $\mathbb{G}_a$-subgroup corresponding to D.

Flexibility vs Infinite Transitivity

Definition

An affine variety X is *flexible* if the tangent space $T_x(X)$ at any smooth point $x \in X^{\mathrm{reg}}$ is generated by velocity vectors to orbits of $\mathbb{G}_a$-subgroups passing through x.

Theorem (A.-Flenner-Kaliman-Kutzschebauch-Zaidenberg'2013)

Let X be an irreducible affine variety of dimension $\geqslant 2$. The following conditions are equivalent:

(a) *the group* $\mathrm{SAut}(X)$ *acts transitively on* X^{reg};

(b) *the group* $\mathrm{SAut}(X)$ *acts infinitely transitively on* X^{reg};

(c) *the variety* X *is flexible.*

Examples of Flexible Varieties

- Suspensions $\mathrm{Susp}(X, f)$ given by $uv = f(x)$ in $\mathbb{A}^2 \times X$ over a flexible variety X;

- Non-degenerate ($\mathbb{C}[X]^\times = \mathbb{C}^\times$) affine toric varieties;

- Non-degenerate horospherical varieties of reductive groups;

- Homogeneous spaces G/F, where G is semisimple and F is reductive;

- Normal affine SL(2)-embeddings;

- Affine cones over flag varieties and del Pezzo surfaces.

Root Subgroups and Demazure Roots

Let X be a variety with an action of a torus T. A $\mathbb{G}_a$-subgroup H in $\mathrm{Aut}(X)$ is called a *root subgroup* if H is normalized in $\mathrm{Aut}(X)$ by the torus T. In this case T acts on H by some character e. Such a character is called a *root* of the T-variety X.

Assume X is toric with acting torus T. What are the roots of X?

Let $p_1, \dots, p_s$ be the primitive lattice vectors on rays of the fan Σ_X.

Definition

A *Demazure root* of the fan Σ_X in a character $e \in M$ such that there exists $1 \leqslant i \leqslant s$ with $\langle e, p_i \rangle = -1$ and $\langle e, p_j \rangle \geqslant 0$ for $j \neq i$.

Theorem (Demazure' 1970)

Let X be a complete or an affine toric varieties. Then root subgroups on X are in bijection with Demazure roots of the fan Σ_X.

Finite generation

Conjecture A. Any generically flexible affine variety X admits a finite collection $\{H_1, \ldots, H_k\}$ of $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(X)$ such that the group $G = \langle H_1, \ldots, H_k \rangle$ acts infinitely transitively on its open orbit.

Idea of the proof:

Step 1. Find $G = \langle H_1, \ldots, H_s \rangle$ that acts on X with an open orbit;
Step 2. Prove that the closure $\overline{G}$ of the subgroup G in $\mathrm{Aut}(X)$ in ind-topology contains 'many other' $\mathbb{G}_a$-subgroups;
Step 3. Prove that $\overline{G}$ acts infinitely transitively on the open orbit;
Step 4. Prove that G acts infinitely transitively on the open orbit.

Step 3 $\Rightarrow$ Step 4 turns out to be true in general.

To Step 2:

Conjecture B. Let X be an affine variety, and $A = \mathbb{C}[X]$ be its structure algebra. Consider the group $G = \langle H_1, \ldots, H_k \rangle$ generated by a finite collection of $\mathbb{G}_a$-subgroups $H_i = \exp(\mathbb{C}D_i) \subset \mathrm{SAut}(X)$, where $D_i \in \mathrm{LND}(A)$, $i = 1, \ldots, k$.

Then the $\mathbb{G}_a$-subgroup $H = \exp(\mathbb{C}D) \subset \mathrm{SAut}(X)$, where $D \in \mathrm{LND}(A)$, is contained in $\overline{G}$ if and only if $D \in \mathrm{Lie}\,\langle D_1, \ldots, D_k \rangle$.

The Toric Case

Theorem (A.-Kuyumzhiyan-Zaidenberg'2019)

For any non-degererate affine toric variety X of dimension at least 2, which is smooth in codimention 2, one can find a finite collection of root subgroups such that the group generated by these subgroups acts infinitely transitively on the smooth locus X^{reg}.

In the proof we use Cox rings and the quotient presentation $\pi\colon \mathbb{A}^s \to X$ by an action of a quasitorus.

Finite Generation for Affine Spaces - I

Theorem (Bodnarchuk'2001)

For any $n \geqslant 3$ and any triangular $h \in \mathrm{Aut}(\mathbb{A}^n) \setminus \mathit{Aff}_n$ we have $\langle \mathit{Aff}_n, h \rangle = \mathit{Tame}_n$.

Corollary

For any $n \geqslant 3$ and any non-affine root subgroup H in $\mathrm{Aut}(\mathbb{A}^n)$ the group $\langle \mathit{Aff}_n, H \rangle$ acts on $\mathbb{A}^n$ infinitely transitively. In particular, one can find $n+2$ root subgroups which generate a subgroup acting infinitely transitively on $\mathbb{A}^n$.

Theorem (A.-Kuyumzhiyan-Zaidenberg'2019)

For any $n \geqslant 3$ one can find three $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ which generate a subgroup acting infinitely transitively on $\mathbb{A}^n$.

Finite Generation for Affine Spaces - II

Let H be the $\mathbb{G}_a$-subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ given by

$$(x_1 + ax_2^2, x_2, \ldots, x_n).$$

Theorem (A.-Kuyumzhiyan-Zaidenberg'2019)

Consider the action of the symmetric group $\mathbb{S}(n)$ on $\mathbb{A}^n$ by permutations. Then for any $n \geqslant 3$ the subgroup

$$G = \langle H, \mathbb{S}(n) \rangle \subset \mathrm{Aut}(\mathbb{A}^n)$$

acts infinitely transitively in $\mathbb{A}^n \setminus \{0\}$.

Finite Generation for Affine Plane - I

Let H_k and R_s be the $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(\mathbb{A}^2)$ given by

$$(x_1 + ax_2^k, x_2) \text{ and } (x_1, x_2 + bx_1^s), \text{ respectively.}$$

Let $G_{k,s} = \langle H_k, R_s \rangle$. We claim that if $ks \neq 2$ then $G_{k,s}$ can not be 2-transitive.
Indeed, if $k = 0$ or $s = 0$, then there are only parallel translations along one coordinate.
If $k = s = 1$, then $G_{1,1}$ is the group $\mathrm{SL}(2)$ and it preserves collinearity.
If $ks > 2$, we take a root of unity ω of degree $ks - 1$ and consider the set

$$S = \{(P, Q) \in \mathbb{A}^2 \times \mathbb{A}^2 \mid P = (x_1, x_2),\ Q = (\omega x_1,\ \omega^s x_2)\}$$

$$P' = (x_1 + ax_2^k, x_2),\ Q' = (\omega x_1 + a(\omega^s x_2)^k,\ \omega^s x_2) = (\omega(x_1 + ax_2^k), \omega^s x_2)$$

$$P'' = (x_1, x_2 + bx_1^s),\ Q'' = (\omega x_1,\ \omega^s x_2 + b(\omega x_1)^s) = (\omega x_1, \omega^s(x_2 + bx_1^s))$$

Finite Generation for Affine Plane - II

Theorem (Lewis-Perry-Straub'2019)

The group $G_{1,2}$ generated by two subgroups

$$(x_1 + ax_2, x_2) \quad \textit{and} \quad (x_1, x_2 + bx_1^2)$$

acts infinitely transitively on $\mathbb{A}^2 \setminus \{0\}$.

The proof is based on a detailed study of the Polydegree Conjecture for plane polynomial automorphisms.

Tits Alternative

Theorem (A.-Zaidenberg'2020)

Let X be an affine toric variety. Consider a collection of one-parameter unipotent subgroups $U_1, \ldots, U_s$ of $Aut(X)$ which are normalized by the torus acting on X. Then the group G generated by $U_1, \ldots, U_s$ verifies the Tits alternative, and, moreover, either is a unipotent algebraic group, or contains a nonabelian free subgroup.

Corollary

If G is infinitely transitive, then G contains a nonabelian free subgroup, and so, is of exponential growth.

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