Toric topology of complexity one

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Standing assumptions

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- $X = X^{2n}$: smooth closed connected orientable manifold;
- $T = T^k$: compact torus;
- $T \circlearrowright X$: effective action
- $0 < \#X^T < \infty$. (quasitoric mfds — yes, moment-angle mfds — no)
- These imply $k \leqslant n$.

Definition

The number $n - k$ is called the complexity of the action.

Equivariant cohomology

- $ET \xrightarrow{T} BT$: universal principal T-bundle;
- $BT \simeq (\mathbb{C}P^{\infty})^k$: classifying space of T;
- $H^*(BT) = R[v_1, \ldots, v_k]$, the ground ring R is either a field or $\mathbb{Z}$;
- $X_T = ET \times_T X$: the Borel construction of X;
- $H^*_T(X) = H^*(X_T)$: equivariant cohomology of a T-manifold X.
- $X_T \xrightarrow{X} BT$: this fibration induces the spectral sequence
- $E_2^{*,*} = H^*(BT) \otimes H^*(X) \Rightarrow H^*_T(X)$.

Equivariant formality

- $X_T \xrightarrow{X} BT$: Serre fibration induces
- $E_2^{*,*} = H^*(BT) \otimes H^*(X) \Rightarrow H_T^*(X)$.

Cohomological equivariant formality in the sense of Goresky–Kottwitz–MacPherson

T-manifold X is called equivariantly formal if $E_*^{*,*}$ collapses at E_2-page.

Eq.formality implies

$$H^*(X) = H_T^*(X) \otimes_{H^*(BT)} R = H_T^*(X)/(l.s.o.p.),$$

where $(l.s.o.p.)$ is a parametric ideal.

Equivariant formality

Simple remark

If $H^{\text{odd}}(X) = 0$ then X is equivariantly formal. Indeed, $E_2^{p,q} = H^p(BT) \otimes H^q(X)$ is zero unless p, q are even, so higher differentials vanish.

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Less obvious

Proposition (Masuda and Panov'06+Franz and Puppe'06)

If $0 < \# X^T < \infty$, then X is equivariantly formal $\Leftrightarrow$ $H^{\text{odd}}(X) = 0$.

One more technical assumption

Technical assumption on the action

Suppose for any subgroup $H \subseteq T$ any connected component of the set

$$X^H = \{x \in X \mid Hx = x\}$$

contains a fixed point. In this case we call the action appropriate.

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Remark (Masuda and Panov'06)

Equivariantly formal action of any complexity is appropriate.

Our interest

Methods of toric topology are well developed in complexity 0.

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Problem posed by Buchstaber and Terzic in 2014

Describe relation between

- topology of manifolds with actions of positive complexity;
- topology and combinatorics of the orbit space.

Extend methods of toric topology to general complexity.

Complexity 0

Locally standard actions of complexity 0

Consider the action of $T = T^n$ on $X = X^{2n}$.

Definition

The action $T \circlearrowright X$ is called locally standard if it is locally modelled by the standard representation of T on $\mathbb{C}^n = \mathbb{R}^{2n}$:

$$(t_1, \ldots, t_n)(z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n),$$

up to automorphism of T.

Since $\mathbb{C}^n / T^n \cong \mathbb{R}^n_{\geqslant 0}$, the orbit space X/T is a manifold with corners.

Equivariant formality in complexity 0

Theorem (Masuda and Panov'06)

If all faces of $P = X/T$ are acyclic, then X is equivariantly formal. Conversely, if X is equivariantly formal manifold with complexity 0 action, then the action is locally standard and all faces of P are acyclic.

In these cases $\operatorname{rk} H^{2j}(X) = h_j(P)$, $H^*_T(X) \cong \mathbb{Z}[P]$, and $H^*(X) \cong \mathbb{Z}[P]/(l.s.o.p.)$,

1. h-numbers are computed from combinatorics of P.
2. $\mathbb{Z}[P]$ is the face ring of the simplicial poset dual to P.

These facts generalize the result of Davis–Januszkiewicz on quasitoric manifolds.

Complexity 1

Complexity 1, examples

Homogeneous examples of complexity one actions

- $T^3 \circlearrowright G_{4,2} = U(4)/U(2) \times U(2)$, the Grassmann mfd. of complex 2-planes in $\mathbb{C}^4$;

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These manifolds are equivariantly formal since $H^{\mathrm{odd}}(X) = 0$.

Complexity 1, examples

Restrictions of complexity 0 actions

- $T^n \circlearrowright X^{2n}$: a quasitoric manifold;
- $T^{n-1} \circlearrowright X^{2n}$: induced action of a subtorus $T^{n-1} \subset T^n$.
- Assume, fixed point set $X^{T^{n-1}}$ is finite.

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Hamiltonian actions

Hamiltonian actions of T^k on symplectic mfd X^{2n} with the moment map $\mu\colon X \to \mathfrak{t}^* \cong \mathbb{R}^k$.

All Hamiltonian actions are equivariantly formal according to Atiyah, Bott, and Kirwan. Topology of Hamiltonian actions of complexity 1 were extensively studied by Karshon and Tolman.

General position of complexity 1 action

- $x \in X^T$: a fixed point
- $\alpha_{x,1}, \ldots, \alpha_{x,n} \in \mathrm{Hom}(T^{n-1}, T^1) \cong \mathbb{Z}^{n-1}$: weights of the tangent representation T_xX i.e.
- $T_xX = V(\alpha_{x,1}) \oplus \cdots \oplus V(\alpha_{x,n})$, where $V(\alpha)$ is the 1-dim. complex (or 2-dim. real) representation given by $tz = \alpha(t) \cdot z$.
- $\alpha_{x,i}$ are defined up to sign.

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Definition

The action $T^{n-1} \circlearrowright X^{2n}$ of complexity 1 is called in general position if $\forall x \in X^T$ any $n-1$ of the weights $\alpha_{x,1}, \ldots, \alpha_{x,n} \in \mathbb{Q}^{n-1}$ are linearly independent.

Actions on $G_{4,2}$, F_3, $\mathbb{H}P^2$, S^6 are in general position.

Local property of complexity 1 actions

Proposition (A.'18)

For an action $T^{n-1} \circlearrowright X^{2n}$ in general position, the orbit space X/T is a closed topological manifold.

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Proposition (Cherepanov'19)

For an action $T^{n-1} \circlearrowright X^{2n}$, the orbit space X/T is a closed topological manifold with corners. Its boundary is nonempty if and only if the action is not in general position.

Global structure of the orbit spaces

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Our study is motivated by

Theorem (Buchstaber and Terzic'14-18)

$$G_{4,2}/T^3 \cong S^5, \quad F_3/T^2 \cong S^4.$$

obtained from the theory of $(2n, k)$-manifolds, developed by Buchstaber and Terzic in 2014-18.

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Both $G_{4,2}$ and F_3 are Hamiltonian. The results of Buchstaber and Terzic are generalized by

Theorem (Karshon and Tolman'18)

If an action $T^{n-1} \circlearrowright X^{2n}$ is Hamiltonian and in general position, then X/T is homeomorphic to S^{n+1}.

Non-Hamiltonian examples

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Proposition (A.'18)

Let $T^n \circlearrowright X^{2n}$ be a quasitoric manifold and the restricted action of $T^{n-1} \subset T^n$ on X is in general position. Then $X/T^{n-1} \cong S^{n+1}$.

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This is related to the result of Arnold'99: $\mathbb{H}P^2/T^1 \cong S^7$, which is a generalization of Kuiper–Massey theorem $\mathbb{C}P^2/\text{conj} \cong S^4$.

General statement for complexity one

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Theorem (A. and Masuda'19)

Let R be a field. If $T^{n-1} \circlearrowright X^{2n}$ is R-equivariantly formal action in general position, then $Q = X/T$ is an R-homology sphere. If, moreover, all stabilizers are connected, then the same holds over $\mathbb{Z}$. If, moreover, $\pi_1(X) = 0$, then $X/T \cong S^{n+1}$.

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This theorem covers the previous results: $G_{4,2}, F_3, \mathbb{H}P^2, S^6$, restricted actions on quasitoric manifolds, and Hamiltonian actions.

Non-general position

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Theorem (A. and Cherepanov'19)

Let L be a finite simplicial complex. Then there exists an equivariantly formal action $T^{n-1} \circlearrowright X_L^{2n}$ such that X_L/T is homotopy equivalent to $\Sigma^3 L$.

The example X_L is a $\mathbb{C}P^1$-bundle over the permutohedral variety (hence projective toric variety, hence Hamiltonian).

Combinatorial structure of complexity 1 actions

Complexity 1

Let $T^{n-1} \circlearrowright X^{2n}$ be an action of complexity 1 in general position. Consider

$$X_0 \subset X_1 \subset \cdots \subset X_{n-2} \subset X_{n-1} = X,$$

where X_i is the union of $\leqslant i$-dimensional orbits, and

$$Q_0 \subset Q_1 \subset \cdots \subset Q_{n-2} \subset Q_{n-1} = Q,$$

$Q_i = X_i/T$: the orbit type filtration of the orbit space $Q = X/T$. Recall that Q is a closed topological $(n+1)$-mfd. We have $\dim Q_i = i$ for $i < n-1$.

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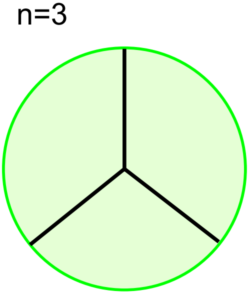
Definition

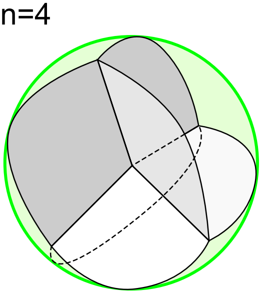
The space $Z = Q_{n-2}$ is called the sponge of the action. It is an $(n-2)$-subspace inside $(n+1)$-dimensional manifold Q. The connected components of $Q_i \backslash Q_{i-1}$, $i \leqslant n-2$ are called the faces of a sponge.

Local structure of a sponge

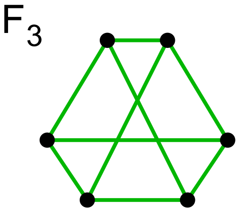
Observation

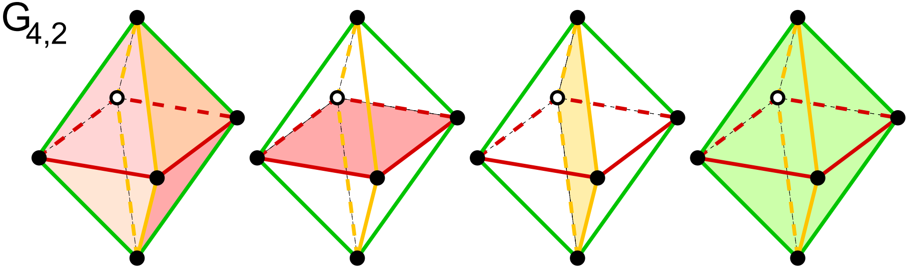
Each point of a sponge Z is locally homeomorphic to $(n-2)$-skeleton of a simple n-dim polytope.





Examples of sponges





Equivariant formality: criterion

Theorem (A. and Masuda'19)

Let $T^{n-1} \circlearrowright X^{2n}$ be an equivariantly formal (over R) action in general position, $Q = X/T$ be its orbit space, and $Z \subset Q$ be the sponge of the action. If $R = \mathbb{Z}$, assume stabilizers are connected. Then

1. The space Q is a homology sphere (i.e. $\forall i \leqslant n$: $\widetilde{H}_i(Q; R) = 0$).
2. Every face F of a sponge Z is acyclic (i.e. $\forall i$: $\widetilde{H}_i(F; R) = 0$);
3. The sponge Z is $(n-3)$-acyclic (i.e. $\forall i \leqslant n-3$: $\widetilde{H}_i(Z; R) = 0$);

If 3 conditions above hold over $R = \mathbb{Z}$, then the action is equivariantly formal over $\mathbb{Z}$.

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Definition

An abstract sponge = space which is locally homeomorphic to $(n-2)$-skeleton of a simple n-polytope. A sponge Z is acyclic if it satisfies conditions (2) and (3) above.

Combinatorics of sponges

Z : an acyclic sponge.

Definition

f_i = the number of i-dimensional faces of Z, and $b = \operatorname{rk} \widetilde{H}_{n-2}(Z)$. The tuple $((f_0, \ldots, f_{n-2}), b)$ is called the extended f-vector of Z.

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Remark $f_0 - f_1 + \cdots + (-1)^{n-2} f_{n-2} = 1 + (-1)^{n-2} b$ since both are equal to $\chi(Z)$. Thus b is expressed by f_i's.

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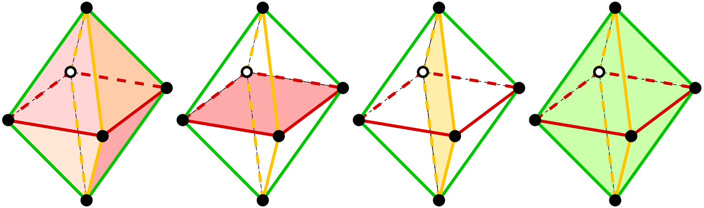
Theorem (A. and Masuda'19)

Let $((f_0, \ldots, f_{n-2}), b)$ be the extended f-vector of the sponge of an equivariantly formal action $T^{n-1} \circlearrowright X^{2n}$ in general position. Then

$$\sum_{i=0}^{n} \beta_{2i}(X) t^{2i} = \sum_{i=0}^{n-2} f_i t^{2n-2i} (1 - t^2)^i + (1 + bt^2)(1 - t^2)^{n-1}.$$

Action of T^3 on $G_{4,2}$.

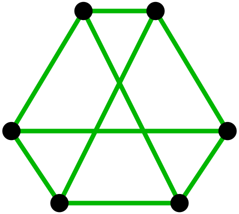


Extended f-vector $= ((6, 12, 11), 4)$. We have

$$\sum_{i=0}^{n} \beta_{2i}(G_{4,2})t^{2i} = 6t^8 + 12t^6(1-t^2) + 11t^4(1-t^2)^2 + (1+4t^2)(1-t^2)^3 =$$

$$1 + t^2 + 2t^4 + t^6 + t^8.$$

Action of T^2 on F_3.



Extended f-vector $= ((6,9),4)$. We have

$$\sum_{i=0}^{n} \beta_{2i}(F_3)t^{2i} = 6t^6 + 9t^4(1-t^2) + (1+4t^2)(1-t^2)^2 = 1 + 2t^2 + 2t^4 + t^6.$$

Combinatorial and topological questions

Definition

Define the h-vector of an acyclic $(n-2)$-sponge by

$$\sum_{i=0}^{n} h_i t^{2i} = \sum_{i=0}^{n-2} f_i t^{2n-2i}(1-t^2)^i + (1+bt^2)(1-t^2)^{n-1}.$$

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Questions

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1. $h_i = h_{n-i}$, "Dehn–Sommerville relations" (work in progress with Cherepanov);
2. $h_i \geqslant 0$, "Lower bound conjecture" (open);
3. Is there a theory of "face rings" of sponges? Can we describe equivariant cohomology of complexity one actions using these rings?

Arbitrary complexity

Actions of any complexity

Let $T^k \circlearrowright X^{2n}$ be an action with isolated fixed points. For a fixed point $x \in X^T$ consider the tangent weights

$$\alpha_{x,1}, \ldots, \alpha_{x,n} \in \mathrm{Hom}(T^k, T^1) \cong \mathbb{Z}^k.$$

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Definition

The action is called in j-general position if, at each $x \in X^T$, any j of the weights $\alpha_{x,1}, \ldots, \alpha_{x,n}$ are linearly independent.

- Isolated fixed points = the action is in 1-general position.
- For GKM-manifolds the action is in 2-general position.
- Complexity 1 action $T^{n-1} \circlearrowright X^{2n}$ is in general position $\Leftrightarrow$ it is in $(n-1)$-general position.

General result on general complexity

Theorem (A. and Masuda'19)

If $T^k \circlearrowright X^{2n}$ is an equivariantly formal action in j-general position, then X/T is $(j+1)$-acyclic (i.e. $\widetilde{H}_i(Q) = 0$ for $i \leqslant j+1$).

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The degree of acyclicity cannot be improved.

Proposition (A. and Cherepanov'19)

For any finite simplicial complex L there exists an equivariantly formal torus action of complexity 1 in j-general position such that X/T is homotopy equivalent to $\Sigma^{j+2}L$.

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- Algebraic combinatorics of sponges.
- Develop general theory for real versions of complexity 1 actions.
- Study relations between sponges and tropical cohomology theory.

Thank you for listening!