T^n-action on the Grassmannians $G_{n,2}$ via hyperplane arrangements

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Complex Grassmann manifolds $G_{n,k} = G_{n,k}(\mathbb{C})$

$G_{n,k}$ – k-dimensional complex subspaces in $\mathbb{C}^n$,

- The coordinate-wise $\mathbb{T}^n$ - action on $\mathbb{C}^n$ induces $\mathbb{T}^n$ - action on $G_{n,k}$.
- This action is not effective — $T^{n-1} = \mathbb{T}^n/\Delta$ acts effectively.
- $d = k(n-k) - (n-1)$ - complexity of T^{n-1}-action;
- $d \geq 2$ for $n \geq k+3$, $k \geq 2$.
- $\mathbb{T}^n$-action extends to $(\mathbb{C}^*)^n$ -action on $G_{n,k}$

<u>Problem:</u> Describe the combinatorial structure and algebraic topology of the orbit space $G_{n,k}/\mathbb{T}^n \cong G_{n,n-k}/\mathbb{T}^n$.

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We describe here the orbit space $G_{n,2}/T^n$ in terms of :

1. "soft" chamber decomposition $L(\mathcal{A}_{n,2})$ for $\Delta_{n,2}$,
   - $\mathcal{A} = \Pi \cup \{x_i = 0,\ 1 \leq i \leq n\} \cup \{x_i = 1,\ 1 \leq i \leq n\}$ - hyperplane arrangement in $\mathbb{R}^n$;
   - $\Pi = \{x_{i_1} + \ldots + x_{i_l} = 1,\ 1 \leq i_1 < \ldots < i_l \leq n,\ 2 \leq l \leq [\frac{n}{2}]\}$;
   - $L(\mathcal{A})$ - face lattice for $\mathcal{A}$;
   - $L(\mathcal{A}_{n,2}) = L(\mathcal{A}) \cap \overset{\circ}{\Delta}_{n,2}$;
2. spaces of parameters F_C for $C \in L(\mathcal{A}_{n,2})$ - parametrize $(\mathbb{C}^*)^n$ - orbits in $\mu_{n,2}^{-1}(C) \subset G_{n,2}$;
3. universal space of parameters $\mathcal{F}$.

Moment map

The Plücker embedding $G_{n,k} \to \mathbb{C}P^{N-1}$, $N = \binom{n}{k}$, is given by

$$L \to P(L) = \big(P_I(A_L),\ I \subset \{1, \dots n\},\ |I| = k\big),$$

$P_I(A_L)$ - Plücker coordinates of L in a fixed basis.

The moment map $\mu_{n,k} : G_{n,k} \to \mathbb{R}^n$ is defined by

$$\mu_{n,k}(L) = \frac{1}{|P(L)|^2} \sum |P_I(A_L)|^2 \Lambda_I, \quad |P(L)|^2 = \sum |P_I(A_L)|^2,$$

where $\Lambda_I \in \mathbb{R}^n$ has 1 at k places and it has 0 at the other $(n-k)$ places, the sum goes over the subsets $I \subset \{1, \dots, n\}$, $|I| = k$.

- $\mathrm{Im}\mu_{n,k} = \mathrm{convexhull}(\Lambda_I) = \Delta_{n,k}$ – hypersimplex.
- $\Delta_{n,k}$ is in the hyperplane $x_1 + \dots + x_n = k$ in $\mathbb{R}^n$, $\dim\Delta_{n,k} = n-1$.
- $\mu_{n,k}$ is $\mathbb{T}^n$-invariant, it unduces the map $\hat{\mu}_{n,k} : G_{n,k}/\mathbb{T}^n \to \Delta_{n,k}$.

T^n-action, moment map and Aut$G_{n,k}$

Lemma

Let $H < AutG_{n,k}$ consists of the elements which commutes with the canonical T^n-action on $G_{n,k}$. Then

- $H = T^{n-1} \rtimes S_n$ *for* $n \neq 2k$;
- $H = \mathbb{Z}_2 \times (T^{n-1} \rtimes S_n)$ *for* $n = 2k$.

Let $f \in \mathrm{Aut}G_{n,k}$ and assume there exists (combinatorial) isomorphism $\bar{f} : \Delta_{n,k} \to \Delta_{n,k}$ such that the diagram commutes:

$$\begin{array}{ccc} G_{n,k} & \xrightarrow{f} & G_{n,k} \\ \downarrow \mu_{n,k} & & \downarrow \mu_{n,k} \\ \Delta_{n,k} & \xrightarrow{\bar{f}} & \Delta_{n,k}. \end{array} \qquad (1)$$

Proposition

Let $H < \mathrm{Aut}G_{n,k}$ consists of those elements which satisfy (1). Then

- $H = T^{n-1} \rtimes S_n$ for $n \neq 2k$; $\quad H = \mathbb{Z}_2 \times (T^{n-1} \rtimes S_n)$ for $n = 2k$.
- $\bar{t} = id_{\Delta_{n,k}}$ for $t \in T^{n-1}$;
- $\bar{\mathfrak{s}}(x_1, \ldots, x_n) = (x_{\mathfrak{s}(1)}, \ldots, x_{\mathfrak{s}(n)})$ for $\mathfrak{s} \in S_n$;
- $\bar{c}_{n,k}(x_1, \ldots, x_n) = (1 - x_1, \ldots, 1 - x_n)$ for $c_{n,k} \in \mathbb{Z}_2$, $n = 2k$ - duality automorphism.

Corollary

- $\hat{\mu}_{n,k}^{-1}(x)$ *is homeomorphic to* $\hat{\mu}_{n,k}^{-1}(\mathfrak{s}(x))$ *for* $x \in \Delta_{n,k}$ *and* $\mathfrak{s} \in S_n$
- $\hat{\mu}_{n,k}^{-1}(x)$ *is homeomorphic to* $\hat{\mu}_{n,k}^{-1}(\mathbf{1} - x)$ *for* $x \in \Delta_{n,k}$*, when* $n = 2k$*.*

Strata on $G_{n,k}$

Let $M_I = \{L \in G_{n,k} \mid P^I(L) \neq 0\}, \quad I \subset \{1, \ldots, n\}, \quad |I| = k$.

- M_I is an open and dense set in $G_{n,k}$ and $G_{n,k} = \bigcup M_I$.
- M_I contains exactly one T^n- fixed point x_I.
- Set $Y_I = G_{n,k} \setminus M_I$.

Let $\sigma \subset \{I, \ I \subset \{1, \ldots, n\}, \ |I| = k\}$ and define the stratum W_σ by

$$W_\sigma = (\cap_{I\in\sigma} M_I) \cap (\cap_{I\notin\sigma} Y_I) \text{ if this intersection is nonempty.}$$

The main stratum is $W = \cap_{I \in \{\binom{n}{k}\}} M_I$ - an open and dense set in $G_{n,k}$.

- $W_\sigma \cap W_{\sigma'} = \emptyset$ for $\sigma \neq \sigma'$,
- W_σ is $(\mathbb{C}^*)^n$ - invariant, $G_{n,k} = \cup_\sigma W_\sigma$
- W_σ are no open, no closed and their geometry is not nice.

Strata on $G_{n,k}$

Lemma

$\mu_{n,k}(W_\sigma) = \overset{\circ}{P}_\sigma, \quad P_\sigma = convhull(\Lambda_I, I \in \sigma)$

Such P_σ is called an admissible polytope

- $\{W_\sigma\}$ coincide with the strata of Gel'fand-Serganova: $W_\sigma = \{L \in G_{n,k} : \mu_{n,k}(\overline{(\mathbb{C}^*)^n \cdot L}) = P_\sigma\}$,
- Any face of an admissible polytope is an admissible polytope.
- $\mu_{n,k}(W) = \overset{\circ}{\Delta}_{n,k}, \quad \mu_{n,k}(\text{fixed point}) = \text{vertex}$.
- $\Delta_{n,k}$ and its faces are admissible polytopes.

Theorem

All points from W_σ have the same stabilizer T_σ ($(\mathbb{C}^*)_\sigma$).

Torus $T^\sigma = T^n / T_\sigma$ acts freely on W_σ.

Moment map decomposes as $\mu_{n,k} : W_\sigma \to W_\sigma/T^\sigma \stackrel{\hat{\mu}_{n,k}}{\to} \overset{\circ}{P}_\sigma$.

Theorem

$\hat{\mu}_{n,k} : W_\sigma/T^\sigma \to \overset{\circ}{P}_\sigma$ is a locally trivial fiber bundle with a fiber an open algebraic manifold F_σ. Thus,

$$W_\sigma/T^\sigma \cong \overset{\circ}{P}_\sigma \times F_\sigma.$$

F_σ – the space of parameter for W_σ;

$$F_\sigma \cong W_\sigma/(\mathbb{C}^*)^\sigma.$$

To summarize: $G_{n,k}/T^n = \cup_\sigma W_\sigma/T^\sigma \cong \cup_\sigma(\overset{\circ}{P}_\sigma \times F_\sigma)$

$$G_{n,k}/T^n = \overline{W/T^{n-1}} \cong \overline{\overset{\circ}{\Delta}_{n,k} \times F}.$$

<u>Goal:</u> Describe P_σ, F_σ and the corresponding compactification $\mathcal{F}$ for F

$\Delta_{n,2} \subset \mathbb{R}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : x_1 + \ldots + x_n = 2\}$; $\dim P_\sigma \leq n-1$, for any σ.

Proposition

If $\dim P_\sigma \leq n-3$ then $P_\sigma \subset \partial\Delta_{n,2}$.

- $\partial\Delta_{n,2} = (\cup_n \Delta^{n-2}) \cup (\cup_n \Delta_{n-1,2})$
- $\mu_{n,k}^{-1}(\partial\Delta_{n,2}) = (\cup_n \mathbb{C}P^{n-2}) \cup (\cup_n G_{n-1,2})$

If $\dim P_\sigma = n-2$ and $P_\sigma \subset \partial\Delta_{n,2}$:

- $P_\sigma = \Delta^{n-2}$ or
- $P_\sigma \subseteq \Delta_{n-1,2}$ is an admissible polytope for $G_{n-1,2}$.

Admissible $(n-2)$- polytopes

Let $\dim P_\sigma = n-2$ and $P_\sigma \cap \overset{\circ}{\Delta}_{n,2} \neq \emptyset$ - interior admissible polytope

Proposition

The interior admissible polytopes of dimension $n-2$ coincide with the polytopes obtained by the intersection with $\Delta_{n,2}$ of the planes

$$\Pi : x_{i_1} + \ldots + x_{i_l} = 1, \;\; 1 \leq i_1 < \ldots < i_l \leq n, \;\; 2 \leq l \leq [\frac{n}{2}].$$

- S_n acts on Π by permutation of coordinates;
- $\Pi_{\{i,j\}}$ - the planes from Π which contain the vertex Λ_{ij};
- $\Pi_{\{i,j\}} : x_{\{i \text{ or } j\}} + x_{l_2} + \ldots + x_{l_s} = 1,\ 2 \leq s \leq [\frac{n}{2}]$;
- $|\Pi_{\{i,j\}}| = 2^{n-2} - 2, \quad S_n \cdot \Pi_{ij} = \Pi$ with stabilizer $S_2 \times S_{n-2}$;

Proposition

The number of irreducible representations for $S_2 \times S_{n-2}$-action on $\Pi_{\{i,j\}}$ is $[\frac{n-2}{2}]$. Their dimensions are:

$$\text{for } n \text{ odd}: \binom{n-2}{l},\ 1 \leq l \leq [\frac{n-2}{2}],$$

$$\text{for } n \text{ even}: \binom{n-2}{l},\ 1 \leq l < [\frac{n-2}{2}] \text{ and } \frac{2}{n-2}\binom{n-2}{\frac{n-2}{2}}.$$

Corollary

An interior $(n-2)$-dimensional polytope has $n_p = p(n-p)$ vertices for $2 \leq p \leq [\frac{n}{2}]$.

Corollary

The number q_p of $(n-2)$- polytopes which have n_p vertices is

$$q_p = \binom{n}{p} \text{ for } n \text{ odd},$$

$$q_p = \binom{n}{p} \text{ for } n \text{ even and } 1 \leq p \leq \frac{n-2}{2},$$

$$q_{\frac{n}{2}} = \frac{1}{2}\binom{n}{\frac{n}{2}} \text{ for } n \text{ even.}$$

Examples.

- $G_{4,2}$ – $\dim P_\sigma = 2$, one S_4-generator, it has 4 vertices, altogether 3 polytopes, $x_1 + x_i = 1$, $i = 2, 3, 4$.
- $G_{5,2}$ – $\dim P_\sigma = 3$, one S_5-generator, it has 6 vertices, altogether 10 polytopes, $x_i + x_j = 1$, $1 \leq i < j \leq 5$
- $G_{6,2}$ – $\dim P_\sigma = 4$, two S_6-generators, they have 8 and 9 vertices, altogether 15 and 10 polytopes respectively (correspond to $S_2 \times S_4$- action on $\mathbb{C}^7$ which has 2 irreducible summands of dimension 4 and 3), $x_i + x_j = 1$, $\quad x_1 + x_i + x_j = 1$, $1 \leq i < j \leq 6$.

Admissible polytopes of dimension $n-1$

Theorem

They are given by $\Delta_{n,2}$ and the closure of the intersections with $\overset{\circ}{\Delta}_{n,2}$ of all collections of the half-spaces of the form

$$x_{i_1} + x_{i_2} + \ldots + x_{i_k} \leq 1, \;\; i_1, \ldots i_k \in \{1, \ldots, n\}, \; 2 \leq k \leq n-2,$$

such that if x_{i_p} and x_{i_q} contribute to the collection then $i_p \neq i_q$, where $1 \leq p, q \leq n-2$.

Examples

- $G_{4,2}$ – $\Delta_{4,2}$ and the half spaces $x_i + x_j \leq 1$, $1 \leq i < l \leq 4$; – $(6, 5)$.

- $G_{5,2} - \Delta_{5,2}$ and the half spaces
  1. $x_i + x_j \leq 1$ — $(10, 9)$.
  2. $x_i + x_j + x_k \leq 1$ — $(10, 7)$.
  3. $x_i + x_j \leq 1$ and $x_p + x_q \leq 1$, $\{i, j\} \cap \{p, q\} = \emptyset$ — $(15, 8)$.
- $G_{6,2} - \Delta_{6,2}$ and ithe half spaces
  1. $x_i + x_j \leq 1$ — $(15, 14)$;
  2. $x_i + x_j + x_k \leq 1$ — $(20, 12)$;
  3. $x_i + x_j + x_k + x_l \leq 1$ — $(15, 9)$
  4. $x_i + x_j \leq 1$ and $x_p + x_q \leq 1$, $\{i, j\} \cap \{p, q\} = \emptyset$ — $(45, 13)$;
  5. $x_i + x_j \leq 1$ and $x_p + x_q + x_s \leq 1$, $\{i, j\} \cap \{p, q, s\} = \emptyset$ — $(60, 11)$.

Space of parameteres F_σ for the strata W_σ

The main stratum W is in the chart M_{12} given by:

$$c_{ij}^{'} z_i w_j = c_{ij} z_j w_i, \;\; 3 \leq i < j \leq n, \tag{2}$$

$$(c_{ij}^{'} : c_{ij}) \in \mathbb{C}P_A^1 = \mathbb{C}P^1 \setminus \{A = \{(1:0), (0:1), (1:1)\}\}.$$

The parameters $(c_{ij} : c_{ij}^{'})$ satisfy the relations:

$$c_{ki}^{'} c_{kj} c_{ij}^{'} = c_{ki} c_{kj}^{'} c_{ij}, \;\; 3 \leq k < i < j \leq n. \tag{3}$$

$$F = W/(\mathbb{C}^{*})^n = \{(c_{ij} : c_{ij}^{'}) \in (\mathbb{C}P_A^1)^N \subset (\mathbb{C}P^1)^N : \; c_{ki}^{'} c_{kj} c_{ij}^{'} = c_{ki} c_{kj}^{'} c_{ij}\},$$

where $N = \binom{n-2}{2}$.

Any straum $W_\sigma \subset M_{12}$ is defined by:

$$P^{1j_2} = 0,\ P^{2i_1} = 0,\ P^{ij} = 0 \ \ 3 \leq i_1, j_1, i, j \leq n, i \neq j.$$

In the local coordinates: $z_{i_1} = w_{j_2} = 0$ and $z_i w_j = z_j w_i$.

$$F_\sigma = \{(c_{ij} : c_{ij}^{'}) \in (\mathbb{C}P^1_B)^l : c_{ki}^{'} c_{kj} c_{ij}^{'} = c_{ki} c_{kj}^{'} c_{ij}\}$$

where $\mathbb{C}P^1_B = \mathbb{C}P^1 \setminus \{B = \{(1:0), (0:1)\}\}$ and $0 \leq l \leq N$.

Proposition

If P_σ is an interior polytope and $\dim P_\sigma = n - 2$ then F_σ is a point.

A universal space of parameters $\mathcal{F}$

We introduced $\mathcal{F}$ in (B-T, MMJ, 2019) to be a compactification of F which realizes:

$$\overline{\overset{\circ}{\Delta}_{n,2} \times F} = G_{n,2}/T^n.$$

$\mathcal{F}$ is axiomatized in (B-T, Mat. Sb, 2019) for $(2n,k)$-manifolds.

- For $G_{5,2}$ we exlicitely described $\mathcal{F}$ in (B-T, MMJ, 2019)
- For general $G_{n,2}$ it is proved (Klemyatin, 2019) that $\mathcal{F}$ is provided by the Chow quotient $G_{n,2}//(\mathbb{C}^*)^n$ by Kapranov.
- Thus, $\mathcal{F}$ is the Grotendick-Knudsen compactification of n-pointed curves of genus 0.

We decribe here $\mathcal{F}$ using representation of F in local charts for $G_{n,2}$ defined by the Plücker coordiantes.

Idea:

- $W_\sigma \subset M_{12}$: $z_{i_1} = w_{j_2} = 0$ and $z_i w_j = z_j w_i$.
- Assign the new space of parameters $\tilde{F}_{\sigma,12}$ to W_σ using (2).
- The assignment $W_\sigma \to \tilde{F}_{\sigma,ij}$ must not depend on a chart $W_\sigma \subset M_{ij}$.
- This determines compactification $\mathcal{F}$ of F in which this assignments should be done.

$$\bar{F} = \{(c_{ij} : c_{ij}^{'}) \in (\mathbb{C}P^1)^N,\ c_{ik}c_{il}^{'}c_{kl} = c_{ik}^{'}c_{il}c_{kl}^{'}\},\ \ N = \binom{n-2}{2}.$$

Theorem

Let $\mathcal{F}$ is obtained by blowing up $\bar{F}$ along the submanifolds $\bar{F}_{ikl} \subset \bar{F}$ defined by

$$\bar{F}_{ikl} : (c_{ik} : c_{ik}^{'}) = (c_{il} : c_{il}^{'}) = (c_{kl} : c_{kl}^{'}) = (1 : 1), \;\; 3 \leq i < k < l \leq n.$$

Then any homeomorphism of F induced by the coordinate change extends to the homeomorphism of $\mathcal{F}$.

Theorem

The space $\mathcal{F}$ is the universal space of parameters for $G_{n,2}$

Example

$G_{5,2}$ — $\mathcal{F}$ is the blow up of $\bar{F} \subset (\mathbb{C}P^1)^3$
$\bar{F} = \{((c_{34} : c_{34}^{'}), (c_{35} : c_{35}^{'}), (c_{45} : c_{45}^{'})) | c_{34}^{'} c_{35} c_{45}^{'} = c_{34} c_{35}^{'} c_{45}\}$
at the point $\bar{F}_{123} = ((1 : 1), (1 : 1), (1 : 1))$ ($\mathcal{F}$ is unique).

Example

$G_{6,2}$ — $\mathcal{F}$ is a blow up of $\bar{F} \subset (\mathbb{C}P^1)^6$ up along:

$$\bar{F}_{345} = \{(((1:1),(1:1),(c_{36}:c_{36}^{'}),(1:1),(c_{46}:c_{46}^{'}),(c_{56}:c_{56}^{'})),$$

$$c_{36}c_{46}^{'} = c_{36}^{'}c_{46},\ c_{36}c_{56}^{'} = c_{36}^{'}c_{56},\ c_{46}c_{56}^{'} = c_{46}^{'}c_{56}\}$$

$$\bar{F}_{346} = \{(((1:1),(c_{35}:c_{35}^{'}),(1:1),(c_{45}:c_{45}^{'}),(1:1),(c_{56}:c_{56}^{'})),$$

$$c_{35}c_{45}^{'} = c_{35}^{'}c_{45},\ c_{35}c_{56}^{'} = c_{35}^{'}c_{56},\ c_{45}c_{56}^{'} = c_{45}^{'}c_{56}\}$$

$$\bar{F}_{356} = (((c_{34}:c_{34}^{'}),(1:1),(1:1),(c_{45}:c_{45}^{'}),(c_{46}:c_{46}^{'}),(1:1)),$$

$$c_{34}c_{45}^{'} = c_{34}^{'}c_{45},\ c_{34}c_{46}^{'} = c_{34}^{'}c_{46},\ c_{45}c_{46}^{'} = c_{45}^{'}c_{46}\}$$

$$\bar{F}_{456} = \{((c_{34}:c_{34}^{'}),(c_{35}:c_{35}^{'}),(c_{36}:c_{36}^{'}),(1:1),(1:1),(1:1))\},$$

$$c_{34}c_{35}^{'} = c_{34}^{'}c_{35},\ c_{34}c_{36}^{'} = c_{34}^{'}c_{36},\ c_{35}^{'}c_{36} = c_{35}c_{36}^{'}\}.$$

At intersection point $S = (1:1)^6$ blowup is not claimed to be unique.

Virtual spaces of parameters

$$W_\sigma \to \tilde{F}_\sigma \subset \mathcal{F} - \text{virtual space of parameters}$$

For $x \in \overset{\circ}{\Delta}_{n,2}$ denote by

$$\tilde{x} = \bigcup_{x \in \overset{\circ}{P}_\sigma} \tilde{F}_\sigma .$$

Theorem - Universality

- $\tilde{x} = \mathcal{F}$ for any $x \in \overset{\circ}{\Delta}_{n,2}$.
- $\tilde{F}_\sigma \cap \tilde{F}_{\sigma'} = \emptyset$ for any $\tilde{F}_\sigma, \tilde{F}_{\sigma'} \subset \tilde{x}$, $x \in \overset{\circ}{\Delta}_{n,2}$.

The chamber decomposition for $\Delta_{n,2}$

Consider the hyperplane arrangement

$$\mathcal{A}: \ \Pi \cup \{x_i = 0, 1 \leq i \leq n\} \cup \{x_i = 1, 1 \leq i \leq n\}.$$

$$\Pi : x_{i_1} + \ldots + x_{i_l} = 1, \ \ 1 \leq i_1 < \ldots < i_l \leq n, \ \ 2 \leq l \leq [\frac{n}{2}].$$

- $L(\mathcal{A})$ – face lattice for the arrangement $\mathcal{A}$
- $L(\mathcal{A}_{n,2}) = L(\mathcal{A}) \cap \overset{\circ}{\Delta}_{n,2}$
- $C \in L(\mathcal{A}_{n,2})$ – "soft" chamber for $\Delta_{n,2}$.

Proposition

The chamber decomposition $L(\mathcal{A}_{n,2})$ coincides with the decomposition of $\overset{\circ}{\Delta}_{n,2}$ given by the intersections of all admissible polytopes .

Chambers and spaces of parameters

- For any $C \in L(\mathcal{A}_{n,2})$ it holds $\hat{\mu}^{-1}(x) \cong \hat{\mu}^{-1}(y) \cong F_C$ – follows from Gel'fand-MacPherson results (Lect. Notes In Math. 1987)
- If $\dim C = n - 1$ then F_C is a smooth manifold (follows from B-T, MMJ, 2019)

Lemma

For any $C \in L(\mathcal{A}_{n,2})$ there exists canonical homeomorphism

$$h_C : \hat{\mu}^{-1}(C) \to C \times F_C.$$

F_C is a compactification F given by the spaces F_σ such that $C \subset \overset{\circ}{P}_\sigma$.

- For $G_{4,2}$ it holds $F_C \cong \mathbb{C}P^1$ for any C,
- In general F_C are not all homeomorphic; easy to verify for $G_{5,2}$.

Chambers and virtual spaces of parameters

Corollary

For any $C \in L(\mathcal{A}_{n,2})$ it holds $\tilde{F}_{\sigma} \cap \tilde{F}_{\bar{\sigma}} = \emptyset$ such that $C \subset P_{\sigma}.P_{\bar{\sigma}}$.

$\mathcal{F}$ - a universal space of parameters: there exist the projections

$$p_{\sigma,12} : \tilde{F}_{\sigma,12} \to F_{\sigma}.$$

Corollary

The union $\mathcal{F} = \bigcup_{C \subset P_{\sigma}} \tilde{F}_{\sigma}$ is a disjoint union for any $C \in L(\mathcal{A}_{n,2})$.
Therefore, it is defined the projection $p_{C,12} : \mathcal{F} \to F_C$ by $p_{C,12}(y) = p_{\sigma,12}(y)$, where $y \in \tilde{F}_{\sigma,12}$.

The orbit space $G_{n,2}/T^n$

$$\mathcal{W}(G_{n,2}) = \bigcup_{C\in L(\mathcal{A}_{n,2})} (C\times F_C) - \text{weighted face lattice for} G_{n,2}$$

$$\overset{\circ}{\Delta}_{n,2} = \bigcup_{C\in L(\mathcal{A}_{n,2})} C \;-\; \text{disjoint}, \quad C\times F_C \cong \hat{\mu}^{-1}(C)$$

$$\hat{\mu}^{-1}(\overset{\circ}{\Delta}_{n,2}) = \bigcup_{C\in L(\mathcal{A}_{n,2})} \hat{\mu}^{-1}(C) \cong \bigcup_{C\in L(\mathcal{A}_{n,2})} C\times F_C.$$

- $S_n \curvearrowright L(\mathcal{A}_{n,2})$ by permuting the coordinates; permutes chambers;
- If $\mathfrak{s}(C) = \hat{C}$ then $\hat{\mu}^{-1}(C) \cong \hat{\mu}^{-1}(\hat{C})$ that is $C\times F_C \cong \hat{C}\times F_{\hat{C}}$;
- It follows $S_n \curvearrowright \mathcal{W}(G_{n,2})$; (reduces the number of its elements)

Altogether,

$$G_{n,2}/T^n \cong \hat{\mu}^{-1}(\overset{\circ}{\Delta}_{n,2}) \cup (n\# G_{n-1,2}/T^{n-1}) \cup (n\#\mathbb{C}P^{n-1}).$$

Propostion

The universal space of parameters $\mathcal{F}_{n-1,k}$ for $G_{n-1,2}(k) \subset G_{n,2}$, $1 \leq k \leq n$ can be obtained as

$$\mathcal{F}_{n-1,k} = \mathcal{F}_{|\{(c_{ij}:c'_{ij}),\ i,j\neq k\}}.$$

Consider the space

$$\mathfrak{P} = \Delta_{n,2} \times \mathcal{F}.$$

and the map

$$G : \mathfrak{P} \to G_{n,2}/T^n, \quad G(x,y) = h_C^{-1}(x, p_{C,12}(y)) \text{ if and only if } x \in C.$$

Theorem

G is a continuous surjection and $G_{n,2}/T^n$ is homeomorphic to the quotient of the space $\mathfrak{P}$ by the map G.