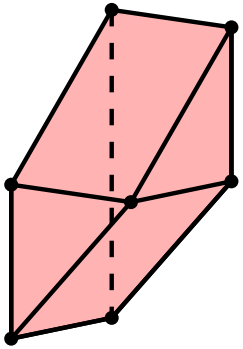
Topology of Singular Toric Varieties

Jongbaek Song
(joint work with A. Bahri, D. Notbohm, S. Sarkar)

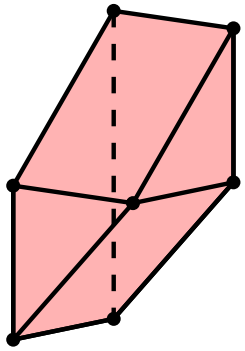
School of Mathematical, KIAS

Workshop on Torus Actions in Topology
Fields Institute, Online Workshop
May 11–15, 2020.

What do you see from this picture?



What do you see from this picture?



Today, we will see this as

1. a non-simple polytope,
2. corresponding singular toric variety.

Qeustions

- P: lattice polytope,
- X_P: (singular) toric variety associated with P.

How much do we know about $H^*(X_P;\mathbb{Z})$?

Qeustions

- P: lattice polytope,
- X_P: (singular) toric variety associated with P.

How much do we know about $H^*(X_P;\mathbb{Z})$?

1. Is $H^{odd}(X_P;\mathbb{Q}) = 0$?
2. Is $H^{odd}(X_P;\mathbb{Z}) = 0$?
3. Is $H^*(X_P;\mathbb{Z})$ torsion free?
4. Under what conditions above statements hold?
5. How to describe its ring structure?

$\vdots$

Qeustions

- P: lattice polytope,
- X_P: (singular) toric variety associated with P.

How much do we know about $H^*(X_P;\mathbb{Z})$?

1. Is $H^{odd}(X_P;\mathbb{Q}) = 0$?
2. Is $H^{odd}(X_P;\mathbb{Z}) = 0$?
3. Is $H^*(X_P;\mathbb{Z})$ torsion free?
4. Under what conditions above statements hold?
5. How to describe its ring structure?

$\vdots$

Remark

- *When P is a Delzant polytope, then we know the complete answer.*
- *When P is a simple polytope, we know partial answers.*
- *For general P, we hardly know the answer.*

Topological construction of a toric variety

1. P: lattice polytope, $\dim P = n$,

$$P = \bigcap_{i=1}^{m} \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i \geq 0\}.$$

   (Two identifications: $\mathbb{R}^n \cong \mathsf{Lie}(T^n)^*$, $\mathbf{a}_i \in \mathsf{Lie}_{\mathbb{Z}}(T^n)$.)

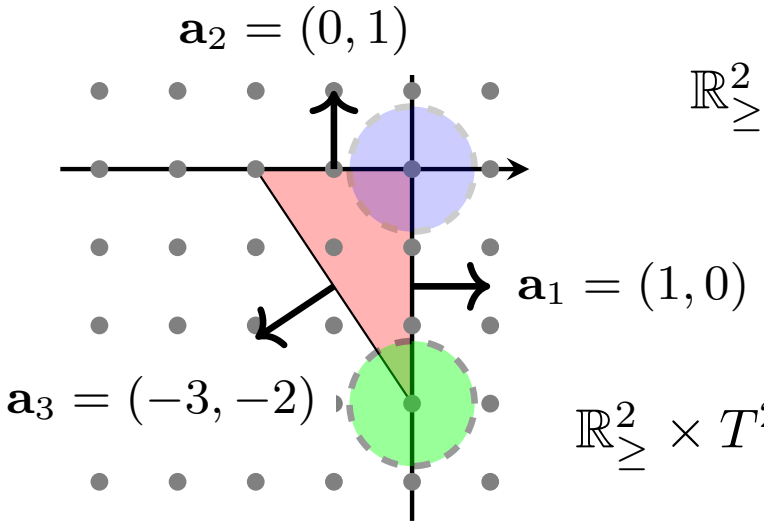
2. $\mathrm{CAT}(P)$: face category of P, i.e., $\begin{cases} \text{object: faces of } P \\ \text{morphism: inclusions} \end{cases}$

3. $\mathscr{D}\colon \mathrm{CAT}(P) \to T^n\text{-}\mathrm{TOP}$ diagram defined by

$$E = F_{i_1} \cap \cdots \cap F_{i_n} \mapsto \mathsf{int}(E) \times T^n / T_{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{\mathbf{i_n}}}.$$

$$X_P := \operatorname{colim} \mathscr{D} \cong P \times T^n / \sim .$$

Example: $\mathbb{C}\mathrm{P}^2_{(1,2,3)}$



$$\mathbb{R}^2_{\geq} \times T^2/_{\sim_{std}} \cong \mathbb{C}^2$$

$$\mathbb{R}^2_{\geq} \times T^2/_{\sim_{std}} \xrightarrow{id \times \theta} \mathbb{R}^2_{\geq} \times T^2/_{\sim} \cong \mathbb{C}^2/G$$

$$G \hookrightarrow T^2 \overset{\theta}{\twoheadrightarrow} T^2$$

$$(t_1, t_2) \overset{\theta}{\mapsto} (t_1 t_2^{-3}, t_2^{-2})$$

Remark

P^n: simple polytope (i.e., neighborhood of each vertex $\cong \mathbb{R}^n_{\geq}$).

1. *If $v = F_{i_1} \cap \cdots \cap F_{i_n}$ and $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n}\}$ form a $\mathbb{Z}$-basis $\forall v \in V(P)$, then X_P is a manifold,*
2. *If $v = F_{i_1} \cap \cdots \cap F_{i_k}$ and $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\}$ linearly indep. $\forall v \in V(P)$, then, X_P is an orbifold.*

Let P and X_P be a simple polytope and the associated toric mfd/orb.

Known results for $H^*(X_P)$

$$\left(H^*(X_P) := H^*(X_P; R), \quad \text{where } R = \begin{cases} \mathbb{Q} & \text{for } X_P = \text{orbifold;} \\ \mathbb{Z} & \text{for } X_P = \text{manifold.} \end{cases} \right)$$

1. $H^{odd}(X_P) = 0$,
2. $H_T^*(X_P) \cong \begin{cases} R[P] & \text{Stanley–Reisner ring of } P \\ \mathrm{PP}_R[\Sigma_P] & \text{the ring of piecewise poly. on } \Sigma_P \\ H^*(\Gamma_P, \alpha) & \text{the cohomology of GKM-graph} \end{cases}$
3. $H^*(X_P) \cong H_T^*(X_P)/H^{>0}(BT) \cong R[x_1, \ldots, x_m]/(\mathcal{I} + \mathcal{J})$.

One reason why manifold and orbifold behave similarly in cohomology:

Around each fixed point in X_P, we have the following information

- $G := \ker(\theta\colon T^n \twoheadrightarrow T^n)$, a finite group,
- $G \curvearrowright \mathbb{C}^n$ linearly, orientation preserving,
- $G \curvearrowright S^{2n-1}$ and $\mathbb{C}^n/G \cong C(S^{2n-1}/G)$

Proposition

$$H^*(S^{2n-1}/G; R) \cong H^*(S^{2n-1}; R),$$

for a field R of characteristic 0 or of characteristic prime to $|G|$.

1. $H^*(S^{2n-1}/G; \mathbb{Q}) \cong H^*(S^{2n-1}; \mathbb{Q})$,
2. $H^i(S^{2n-1}/G; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, 2n-1 \\ 0 \text{ or } |G|\text{-torsion} & 1 \leq i \leq 2n-2. \end{cases}$

What happens for the case where P is not a simple polytope?

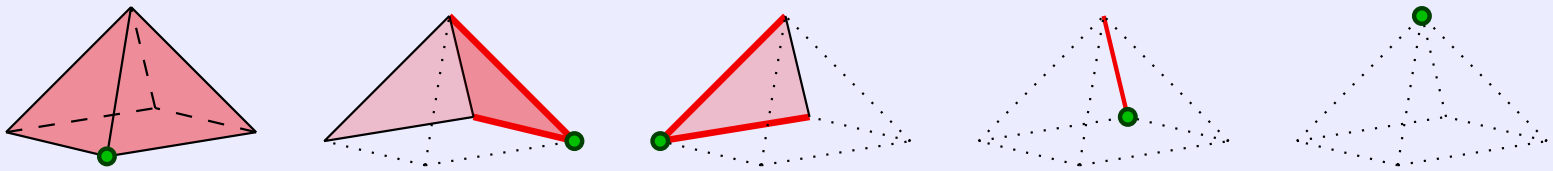
Recall from previous pages that a "simple" vertex is something we can handle cohomologically.

What happens for the case where P is not a simple polytope?

Recall from previous pages that a "simple" vertex is something we can handle cohomologically. Hence, our strategy is...

Definition (by an example)

A *retraction sequence* of P is a sequence of polytopal complexes constructed as follows:

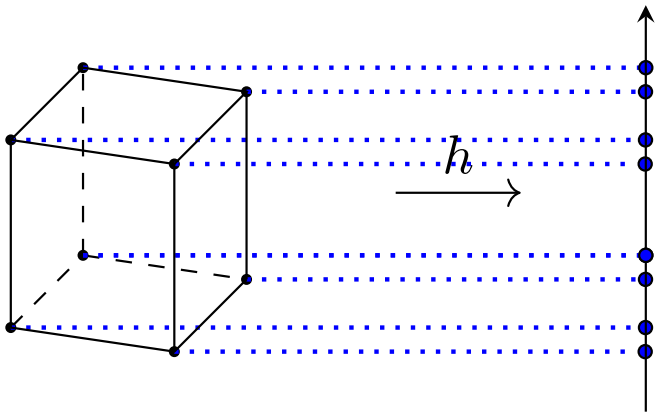


$P = P_1$ $\quad$ P_2 $\quad$ P_3 $\quad$ P_4 $\quad$ P_5

Several remarks

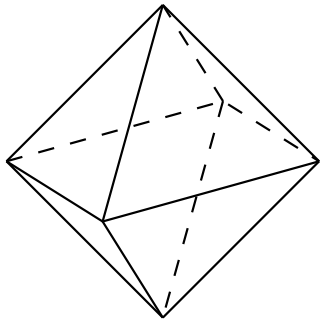
1. Every simple polytope has at lease one retraction sequence.
   (One way to obtain this is to use a height function.)

e.g.

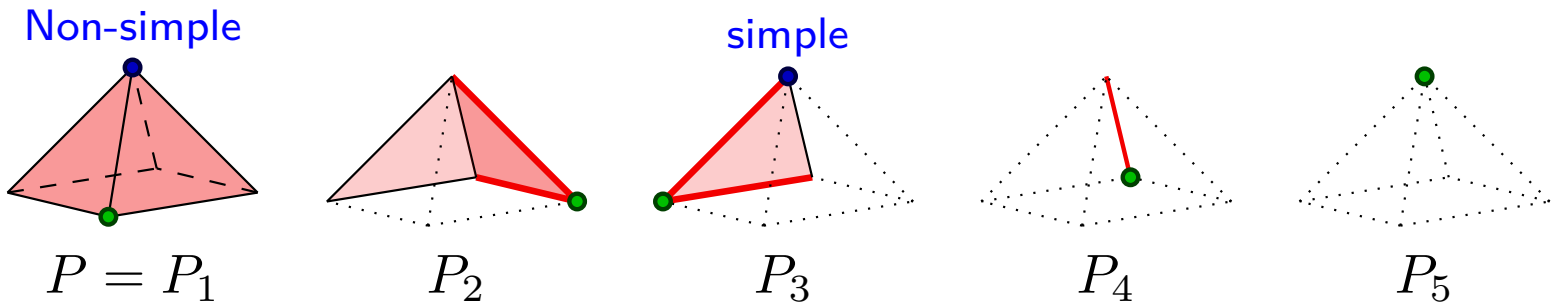


2. Not every convex polytope has a retraction sequence.

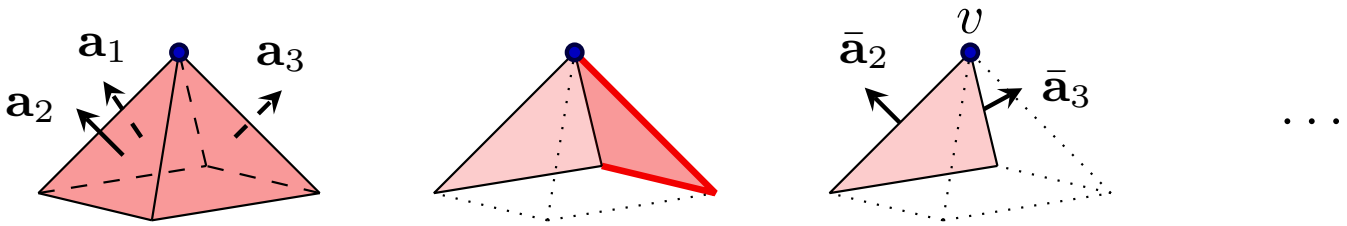
e.g.



3 However, still wide class of convex polytope has a retraction sequence. This is possible mainly because..



4. We can also measure the "smoothness" of each vertex as follows.



$$\mathbb{Z}^3 \xrightarrow{\rho} \mathbb{Z}^2 \cong \mathbb{Z}^3 / \langle \mathbf{a}_1 \rangle$$

$$\mathbf{a}_i \mapsto \rho(\mathbf{a}_i),\ \bar{\mathbf{a}}_i := \mathrm{prim}(\rho(\mathbf{a}_i))$$

Main result

Theorem (Bahri–Notbohm–Sarkar–S, '19)

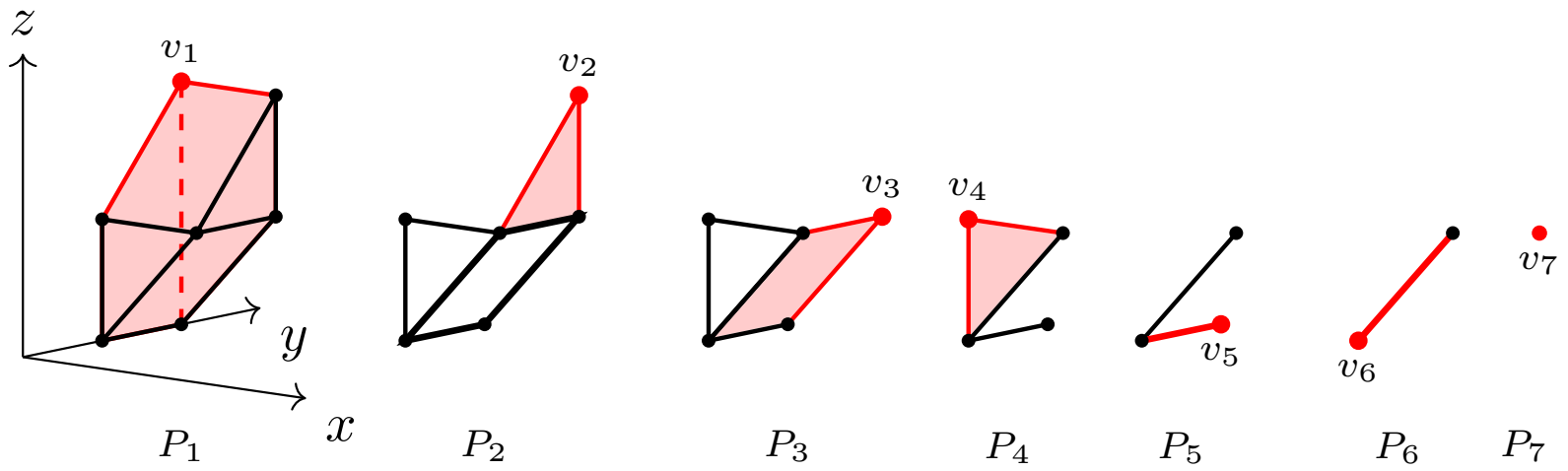
Let P and X_P be as above.

1. *If P admits a retraction sequence. Then, $H^{odd}(X_P;\mathbb{Q}) = 0$.*
2. *Furthermore, if a retraction sequence can be given by a sequence of smooth vertices, then $H^*(X_P;\mathbb{Z})$ is torsion free and $H^{odd}(X_P;\mathbb{Z}) = 0$.*

- We will discuss a more general statement later.
- Some possible applications will also be discussed.

Example: Gelfand–Zetlin polytope

Example: 3-dimensional GZ-polytope has a smooth retraction sequence!



1. $H^{odd}(X_{GZ(3)};\mathbb{Z}) = 0$,
2. $H^*_{T^3}(X_{GZ(3)};\mathbb{Z}) \cong \mathrm{PP}[\Sigma_{GZ(3)}]$,
3. $H^*(X_{GZ(3)};\mathbb{Z}) \cong \mathrm{PP}[\Sigma_{GZ(3)}]/H^{>0}(BT^3;\mathbb{Z})$.

A brief sketch of the proof

1. $\pi\colon X_P \to P$ the orbit map with respect to T^n-action.
2. A retaction sequence induces a filtration

$$\begin{tikzcd}
\pi^{-1}(P_\ell) \arrow[r, hook] \arrow[d] & \pi^{-1}(P_{\ell-1}) \arrow[r] \arrow[d] & \cdots \arrow[r] & \pi^{-1}(P_2) \arrow[r, hook] \arrow[d] & \pi^{-1}(P) = X_P \arrow[d] \\
P_\ell \arrow[r, hook] & P_{\ell-1} \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r] & P_1 = P
\end{tikzcd}$$

3. Each square of the filtration above has the following iterated cofibration structure:

$$\begin{tikzcd}
 & S^{k_{i+1}}/G_{i+1} \arrow[r, hook] \arrow[dl] & C(S^{k_{i+1}}/G_{i+1}) \arrow[dr] & S^{k_i}/G_i \arrow[r, hook] \arrow[dl] & C(S^{k_i}/G_i) \arrow[dr] & \\
\cdots \pi^{-1}(P_{i+1}) \arrow[rr, hook] \arrow[d] & & \pi^{-1}(P_i) \arrow[rr, hook] \arrow[d] & & \pi^{-1}(P_{i-1}) \cdots \arrow[d] \\
\cdots P_i \arrow[rr, hook] & & P_i \arrow[rr, hook] & & P_{i-1} \cdots
\end{tikzcd}$$

4. Take a long exact sequence of
$(X_{i-1} := \pi^{-1}(P_{i-1}),\ X_i := \pi^{-1}(P_i))$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{j+1}(X_{i-1}, X_i) \arrow[r] \arrow[d, "\cong"] & H_j(X_i) \arrow[r] & H_j(X_{i-1}) \arrow[r] & H_j(X_{i-1}, X_i) \arrow[r] \arrow[d, "\cong"] & \cdots \\
& \widetilde{H}_j(S^{k_{i-1}}/G_{i-1}) & & & \widetilde{H}_{j-1}(S^{k_i}/G_i) &
\end{tikzcd}$$

5. Use the induction.

More general framework: $\mathbf{q}$-CW complex

Definition

- $\mathbf{q}$-disk: D^n/G, where $G < O(n)$.
- $\mathbf{q}$-CW complex is defined inductively:

  1. $X_0 =$ discrete set of 0-dimensional $\mathbf{q}$-cells,
  2. $X_n = X_{n-1} \cup_{\{f_\alpha\}} \{D^n/G_\alpha\}$, where

$$f_\alpha \colon \partial D^n/G_\alpha \cong S^{n-1}/G_\alpha \to X_{n-1}.$$

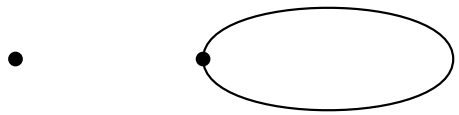
Definition

Given a $\mathbf{q}$-CW complex X, a building sequence is a sequence $\{Y_i\}_{i=1}^{\ell}$ of $\mathbf{q}$-CW subcomplexes of X such that

$$Y_i - Y_{i-1} \simeq \mathrm{int}(D^{k_i}/G_i)$$

for some k_i dimensional $\mathbf{q}$-disk D^{k_i}/G_i.

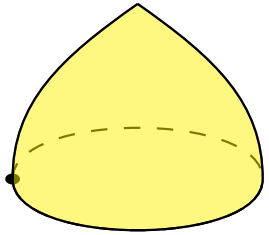
Example: $\mathbb{C}\mathrm{P}^1_{p.q}$



Y_1

$Y_2 = Y_1 \cup_{f_1} D^1$

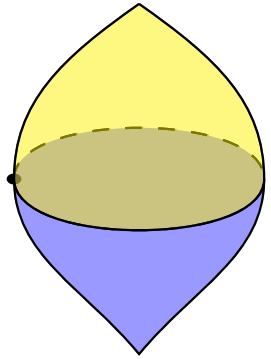
$\simeq S^1$



$Y_3 = Y_2 \cup_{f_2} D^2/\mathbb{Z}_p$

$f_2 \colon S^1/\mathbb{Z}_p \to S^1$

$[e^{2\pi i x}] \mapsto e^{2\pi i p x}$



$Y_4 = Y_3 \cup_{f_3} D^2/\mathbb{Z}_q$

$f_3 \colon S^1/\mathbb{Z}_q \to S^1 \subset Y_3$

$[e^{2\pi i y}] \mapsto e^{2\pi i q y}$

Remark

Building sequence $\{Y_i\}_{i=1}^{\ell}$ is an iterated cofiber seqeunces:

$$\partial D^{k_i}/G_i \xrightarrow{f} Y_{i-1} \hookrightarrow Y_i = \mathrm{cone}(f)$$

Main Results

Theorem (Bahri–Notbohm–Sarkar–S, '19)

Let X be a **q**-*CW complex with no odd dimensional* **q**-*cells and p a prime number. If X has a building sequence $\{Y_i\}_{i=1}^{\ell}$ s.t.*

$$\gcd\{p, |G_i|\} = 1, \quad i = 1, \ldots, \ell,$$

then $H_*(X;\mathbb{Z})$ has no p-torsion and $H_{odd}(X;\mathbb{Z}_p) = 0$.

Corollary (Bahri–Notbohm–Sarkar–S, '19)

Let X be a **q**-*CW complex with no odd dimensional* **q**-*cells. If for each prime p, X has a building sequence $\{Y_i^{(p)}\}_{i=1}^{\ell}$ s.t.*

$$\gcd\{p, |G_i|\} = 1, \quad i = 1, \ldots, \ell,$$

then $H^*(X;\mathbb{Z})$ is torsion free and $H^{odd}(X;\mathbb{Z}) = 0$.

Application in generalized cohomology theories

Theorem [Harada–Holm–Ray–Williams, '16]

- $\mathbb{P}(\chi)$: a divisive weighted projective space.
  i.e. $\chi = (\chi_0, \ldots, \chi_n)$ with $\chi_{i-1} | \chi_i$ for $i = 1, \ldots, n$.
- E_T^*: T-equiv. generalized cohom. theory,
  e.g. $E_T^* = H_T^*,\ K_T^*,\ MU_T^*$.

$$\Rightarrow\ E_T^*(\mathbb{P}(\chi)) \cong \lim(\mathcal{EV}\colon \mathrm{CAT}^{op}(\Sigma_\chi) \to \mathrm{GCALG}_E)$$

$$\cong \begin{cases} \mathrm{P_H}(\Sigma_\chi) & \text{for } E = H, \\ \mathrm{P_K}(\Sigma_\chi) & \text{for } E = K, \\ \mathrm{P_{MU}}(\Sigma_\chi) & \text{for } E = MU. \end{cases}$$

Remarks

- The theorem above is obtained by "generalized GKM-theory" [Harada–Henriques–Holm, '06]
- Their theory holds for any G-space X with a G-equivariant stratificaion $X_1 \subset X_2 \subset \cdots \subset X_\ell$ such that
  1. $X_i/X_{i-1} \simeq \mathrm{Th}(\rho_i)$ with attaching map $\phi_i \colon S(\rho_i) \to X_{i-1}$.
  2. $\rho_i \cong \bigoplus_{j \leq i} (\rho_{i,j} \colon V_{i,j} \to F_i)$
  3. $\exists\ f_{i,j} \colon F_i \to F_j$ such that $(f_{i,j} \circ \rho_{i,j})|_{S(\rho_{i,j})} = \phi_i|_{S(\rho_{i,j})}$.
  4. $e_G(\rho_{i,j})$'s are not divisors of zero in $E_G^*(F_i)$ for $j < i$, and pairwise relatively prime.

Proposition

Let P be a lattice polytope having a smooth retraction sequence. Then, the associated toric variety X_P is equipped with a T-equivariant stratificaion satisfying the above four assumptions.

Theorem

Let P and X_P be as in Proposition above. Then,

$$\begin{aligned} E_T^*(X_P) &\cong \lim(\mathcal{EV}\colon \mathrm{CAT}^{op}(\Sigma_P) \to \mathrm{GCALG}_E) \\ &\cong \begin{cases} \mathrm{P_H}(\Sigma_P) & \textit{for } E = H, \\ \mathrm{P_K}(\Sigma_P) & \textit{for } E = K, \\ \mathrm{P_{MU}}(\Sigma_P) & \textit{for } E = MU. \end{cases} \end{aligned}$$

Thank you for your attention.