Characteristic classes of singular spaces with group action

Andrzej Weber

Institute of Mathematics
University of Warsaw

Joint work with: Laszló Fehér and Richárd Rimányi

The main object of interest

- X an algebraic variety over $\mathbb{C}$, possibly singular, embedded into a nonsingular ambient space M
- G an algebraic linear connected group acting on X and the action extends to the ambient space M

We assume that M is the union of a finite number of G-orbits. This assumption may be relaxed (when the stratification by orbit types behaves well).

The main example: Flag varieties, Grassmannians

$G = B_n$ – upper triangular $n \times n$ matrices, Borel subgroup

$M = GL_n(\mathbb{C})/B_n$ or $GL_n(\mathbb{C})/P$, where $B_n \subset P$

X is a Schubert variety, i.e. the closure of a B-orbit.

More examples

- Matrix varieties:
  $G = GL_k(\mathbb{C}) \times GL_n(\mathbb{C})$, $M = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$
  $$X = \Sigma_r = \{A \in M \ : \ rk(A) \leq r\}$$
  closure of the fixed rank locus.
- Matrix Schubert varieties:
  $G = GL_k(\mathbb{C}) \times B_n(\mathbb{C})$, $M = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$
  X the closure of a G orbit.
- Quiver varieties: e.g. the A_n case $\quad \bullet \to \bullet \to \cdots \to \bullet \to \bullet$
  $$G = \prod_{i=1}^{n} GL_{k_i}(\mathbb{C}), \qquad M = \prod_{i=2}^{n} \mathrm{Hom}(\mathbb{C}_{k_{i-1}}, \mathbb{C}_{k_i})$$
- More classical examples:
  The space of complete quadrics, i.e. the wonderful compactification of $SL_n(\mathbb{C})/SO_n(\mathbb{C})$,
  X – the closure of a B_n orbit
- ... spherical varieties with Borel group action.

The invariants

What kind of homological invariants can singular varieties have?

- Equivariant fundamental class in $H^*_G(M, \mathbb{Q})$
- Equivariant Chern-Schwartz-MacPherson class in $H^*_G(M, \mathbb{Q})$
- The Hirzebruch class $\hat{H}^*_G(M, \mathbb{Q})[y]$.

  It is universal among ,,motivic characteristic classes".
- Classes in equivariant K-theory: $K_G(M)$ – algebraic or topological K-theory
- Borisov-Libgober elliptic classes.

  It defined only for a varieties with mild singularities.

Other complex-oriented cohomology theories

Problems with definition of fundamental classes:

- There are at least three notions of the fundamental class in K-theory
  - Image of $1\!\!1_{\tilde{X}}$ obtained from a resolution $\tilde{X} \to X$
  - $[\mathcal{O}_X]$
  - ,,motivic fundamental class"
- General cohomology theory: for a resolution of singularities

$$\widetilde{X} \to X \hookrightarrow M$$

the image of

$$1\!\!1_{\tilde{X}} \in h^*_G(X) \longrightarrow h^*_G(M)$$

does depend on the resolution of singularities.

Motivic characteristic classes - definition

A notion of a characteristic class for vector bundles $E \to X$

$$\varphi(E) \in h_{\mathbb{T}}(M).$$

allows to define characteristic classes of manifolds. We are looking for an extension of φ for singular sub-varieties of a smooth ambient space, possibly not closed.

We demand that

1. If $X \subset M$ is smooth and closed, then $\Phi(X, M) = \iota_*(\varphi(TX))$
2. If $X \subset M$, $Y \subset X$ is closed, then
$$\Phi(X, M) = \Phi(Y, M) + \Phi(X - Y, M)$$
3. If $f : M_1 \to M_2$ is a proper map, $X_i \subset M_i$ and $f_{|X_1}$ is an isomorphism on the image X_2, then
$$f_*\Phi(X_1, M_1) = \Phi(X_2, M_2)\,.$$

If an extension Φ of φ exists then it is unique.

Such an invariant of a singular variety is called ,,motivic".

Motivic classes in usual cohomology and in the K-theory

- Chern-Schwartz-MacPherson class: according to the original definition the it lives in **homology** of the singular variety.
  - We consider $\mathrm{c}^{\mathrm{sm}}(X, M) \in H^*_G(M)$ in the cohomology of the ambient space.
- The Hirzebruch class for singular varieties

$$td_y(X, M) = td(M) \cdot ch\big(\sum [\Lambda^k T^* X]\, y^k\big) \in \hat{H}_G(M)[y].$$

- Motivic Chern class (Brasselet-Schürmann-Yokura) $\mathrm{mC}(X, M) \in K_G(M)[y]$, the extension of

$$\lambda_y(T^* X) = \sum [\Lambda^k T^* X]\, y^k.$$

In addition: characteristic classes in elliptic theory.

- For (at worst) *Kawamata log-terminal singularities* Borisov and Libgober define elliptic classes in $\hat{H}^*_G(M)[[q, z]]$.

The elliptic classes are do not admit motivic extension.

References

- Nonequivariant classes
  - csm: M.H. Schwartz (1965), R. MacPherson (1974),
  - a different approach P. Aluffi ($\sim$ 2000)
  - Hirzebruch classes, mC:
    J.P. Brasselet, J. Schürmann, S. Yokura (2010)
  - elliptic: L. Borisov, A. Libgober (2003)
- Equivariant classes
  - csm: T. Ohmoto (2006), AW - localization (2012)
  - Hirzebruch classes: AW (2016)
  - mC: Fehér-Rimányi-AW, Aluffi-Mihalcea-Schürmann-Su (2019)
  - Elliptic: R. Waelder (2008)
  - $\vdots$
  - Okounkov's stable envelopes

Main tool: localization theorem for torus action

Let $M_1 \xrightarrow{f} M_2$ be a proper equivariant map of smooth $\mathbb{T}$-varieties.
Define a modified restriction map:

$$res_M : h_{\mathbb{T}}(M) \longrightarrow S^{-1}h_{\mathbb{T}}(M^{\mathbb{T}})$$

$$\alpha \mapsto \frac{\alpha_{M^{\mathbb{T}}}}{e_h(\nu_{M/M^{\mathbb{T}}})}$$

where $S \subset h_{\mathbb{T}}(pt)$ is the multiplicative system generated by the Chern classes of nontrivial line representation,
$e_h(\nu_{M/M^{\mathbb{T}}})$ is the Euler class of the normal bundle.

Theorem (Lefschetz-Riemann-Roch)

The diagram

$$\begin{array}{ccc} h_{\mathbb{T}}(M_1) & \xrightarrow{res_{M_1}} & S^{-1}h_{\mathbb{T}}(M_1^{\mathbb{T}}) \\ \downarrow f_* & & \downarrow f_*^{\mathbb{T}} \\ h_{\mathbb{T}}(M_2) & \xrightarrow{res_{M_2}} & S^{-1}h_{\mathbb{T}}(M_2^{\mathbb{T}}) \end{array}$$

commutes.

The case of isolated fixed points

Lefschetz-Riemann-Roch

In particular, when $x \in M_2$ is isolated and $f^{-1}(x)$ is discrete, then

$$\frac{f_*(\alpha)}{e_h(T_x M_2)} = \sum_{y \in f^{-1}(x)} \frac{\alpha_{|y}}{e_h(T_y M_1)} \in S^{-1} h_{\mathbb{T}}(pt)$$

- The special case: $M \to pt$, $h = H^*(-)$: Atiyah-Bott-Berline-Vergne integration formula.

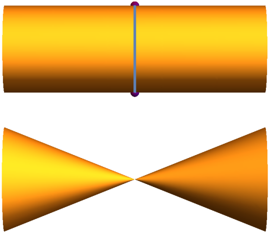
$$\int_M \alpha = \sum_{y \in M^{\mathbb{T}}} \frac{\alpha_{|y}}{e(T_y M)} \in H_{\mathbb{T}}(pt)$$

Example of computation:
The space of symmetric 2×2 matrices with GL_2 action

- The maximal torus action:

$$\begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} x & y \\ y & z \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} s^2x & sty \\ sty & t^2z \end{pmatrix}$$

- The weights of the action: $2s\,,\ s+t\,,\ 2t$.
- Let X be the set of rank 1 matrices $\{xz - y^2 = 0\} \setminus \{(0,0,0)\}$.
- Let $\tilde{M}$ be the blow-up of M at 0.



Computation of $\mathrm{mC}(X, M)$ by Lefschetz-Riemann-Roch.

- The exceptional locus is $\mathbb{P}^1$. In the neighbourhood of the fixed point $[1:0:0]$ there are variables: x, $y' = y/x$, $z' = z/x$.
- The tangent weights are $2s$, $s + t - 2s = t - s$, $2t - 2s = 2(t - s)$.
- The equation of $\pi^{-1}(X)$: $\{z' - {y'}^2 = 0,\ x \neq 0\}$. Normal weight is $2(t - s)$.
- Chern Schwartz MacPherson class:
  $\frac{c^{sm}(X,M)}{e(M)} = \frac{1}{2s}\,\frac{1+t-s}{t-s} + \frac{1}{2t}\,\frac{1+s-t}{s-t} = \dots = \frac{2(1+s+t)(t+s)}{2s\,2t\,(s+t)}$
- Motivic Chern class:
  $\frac{mC(X,M)}{e_K(M)} = \frac{(1+y)s^{-2}}{1-s^{-2}}\,\frac{1+yt^{-1}s}{1-t^{-1}s} + \frac{(1+y)t^{-2}}{1-t^{-2}}\,\frac{1+yts^{-1}}{1-ts^{-1}} = \dots$
- (after inversion of variables $s := s^{-1}$, $t := t^{-1}$)

$$\mathrm{mC}(X, M) = (1 + y)(1 - st)(s^2 + st + t^2 - s^2t^2 + yst(1 + st))$$

- Motivic fundamental class in K-theory: $y = 0$

$$\mathrm{mC}_{y=0}(\overline{X}, M) = \mathrm{mC}_{y=0}(X, M) + \mathrm{mC}_{y=0}(\{0\}, M) = 1 - s^2t^2$$

Goal

The definition of characteristic classes of singular varieties involve resolution of singularities and understanding the push-forward map. In general it is a serious obstacle and involves enormous computational problems.

- Find properties of characteristic classes which allow to compute them without resolving singularities.
- Relate the characteristic classes of varieties coming from representation theory to the underlying algebra. Find a structure governing the formulas.
- Technically: Assuming that a connected algebraic group is acting, take an advantage of the equivariant theory, in particular apply localization theorem for torus action. If the fixed point set $X^{\mathbb{T}}$ is finite, reduce the computation to the algebra of rational functions.

Our setup

- Assume that M is a finite sum of G-orbits.
- Any equivariant cohomology class is determined by the set of restrictions to the orbits.
- In general it is necessary to assume the Atiyah condition (perfect stratification).
- If $G/G_x \simeq \Omega \subset M$ is an orbit, then for $\alpha \in K_G(M)$ we consider the restriction

  $$\alpha_{|\Omega} \in K^*_G(\Omega) \simeq K_{G_x}(pt) \subset K_{\mathbb{T}_x}(pt) = \mathrm{R}(\mathbb{T}_x)$$

  where $\mathbb{T}_\Omega = \mathbb{T}_x \subset G_x$ is a maximal torus.
- If M is a vector space, e.g. when we study the matrix Schubert varieties, then the restriction map

  $$K_G(M) = \mathrm{R}(G) \subset \mathrm{R}(\mathbb{T}) \longrightarrow \mathrm{R}(\mathbb{T}_\Omega)$$

  is given by a substitution in the Laurent polynomial algebra.

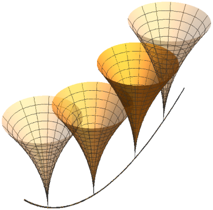
Theorem ([Fehér-Rimányi-AW] Characterization of mC)

Suppose M has finitely many G-orbits and stabilizers are connected. Then the equivariant mC classes of the orbits are characterized by the conditions:

- *(divisibility condition) for any Θ the restriction $\mathrm{mC}(\Omega, M)_{|\Theta}$ is divisible by $c_K(T\Theta)$ in $K_G(\Theta)[y]$.*
- *(normalization condition) $\mathrm{mC}(\Omega, M)_{|\Omega} = e_K(\nu_\Omega)c_K(T\Omega)$;*
- *(smallness condition) there is an inclusion of* **Newton polytopes** *in $K_{\mathbb{T}_\Theta}(pt) \simeq \mathbb{Z}[t_1^\pm, t_2^\pm, \ldots, t_{rk\mathbb{T}_\Theta}^\pm]$*

$$\mathcal{N}(\mathrm{mC}(\Omega, M)_{|\Theta}) \subseteq \mathcal{N}(\mathrm{mC}(\Theta, M)_{|\Theta}) \setminus \{0\}$$

for $\Theta \neq \Omega$.



Smallness

For csm classes [Fehér-Rimanyi]:

$$\deg(c^{sm}(\Omega, M)_{|\Theta}) < \deg(c^{sm}(\Theta, M)_{|\Theta}) \text{ for } \Theta \neq \Omega.$$

For mC classes: we compare

$$\mathrm{mC}(\Omega, M)_{|\Theta} \qquad \text{and} \qquad \mathrm{mC}(\Theta, M)_{|\Theta} = e_K(\nu_\Theta) c_K(T\Theta)$$

in $K_G(\Theta) \hookrightarrow K_{\mathbb{T}_\Theta}(pt)$, where $\mathbb{T}_\Theta$ is the maximal torus in the stabilizer G_Θ.

- We treat them as Laurent polynomials. The size of Newton polytopes plays the role of the degree

$$\mathcal{N}(\mathrm{mC}(\Omega, M)_{|\Theta}) \subseteq \mathcal{N}(\mathrm{mC}(\Theta, M)_{|\Theta}) \setminus \{0\} \qquad \subset \quad \mathbb{T}^\vee_\Theta \otimes \mathbb{R} = \mathfrak{t}^*_{\Theta,\mathbb{R}}.$$

- (*) We have more detailed information

$$\mathcal{N}(\mathrm{mC}(\Omega \cap Slice, Slice)_{|\Theta}) \subseteq \mathcal{N}(e_K(\nu_\Theta)) \setminus \{0\}$$

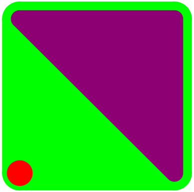
- The inclusion of Newton polytopes is proven with a help of Białynicki-Birula decoposition for $\mathbb{T} = \mathbb{C}^*$.

Example: $\mathbb{C}^2$ with GL_2 action

- Let the characters of the maximal torus be s^{-1}, t^{-1}:
- Let $\Omega = \mathbb{C}^2 - \{0\}, \quad \Theta = \{0\}$:

■ $mC(\Theta, \mathbb{C}^2)_{|\Theta} = (1-s)(1-t) = 1 - s - t + st$

■ $mC(\Omega, \mathbb{C}^2)_{|\Theta} = (1+ys)(1+yt) - (1-s)(1-t) =$
$= (y+1)s + (y+1)t + (y^2-1)st$



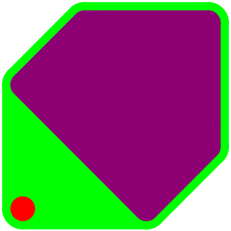
Quadric $\Omega = \{xz - y^2\}$ in $\mathbb{C}^3$, $\quad \Theta = \{0\}$

- $\mathrm{mC}(\Omega, \mathbb{C}^3)_{|\Theta} = (1+y)(1-st)(s^2+st+t^2-s^2t^2+yst+ys^2t^2)$

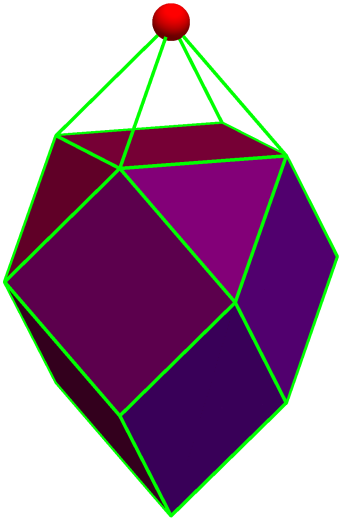
- $\mathcal{N}(\mathrm{mC}(\Omega, \mathbb{C}^3)_{|\Theta}) = conv\{20,\ 11,\ 31,\ 02,\ 22,\ 13,\ 33\}$
(Minkowski sum of an interval and a triangle intervals)

- $\mathrm{mC}(\Theta, \mathbb{C}^3)_{|\Theta} = (1-st)(1-s^2)(1-t^2)$

- $\mathcal{N}(\mathrm{mC}(\Theta, \mathbb{C}^3)_{|\Theta}) = conv\{00,\ 20,\ 11,\ 31,\ 02,\ 22,\ 13,\ 33\}$
(Minkowski sum of 3 intervals)



Newton polytope of $\mathrm{mC}(\textit{open orbit}, \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2))$



Relation with stable envelopes

- The smallness condition is a version of Okounkov axiom for stable envelopes. These are some characteristic classes associated to the Białynicki-Birula cells (a.k.a attractive sets) for **complex** symplectic $\mathbb{T}$-manifolds.

- In the case of T^*G/B the mC classes coincide with the stable envelopes (for a particular slope). For general T^*M one needs to assume that Białynicki-Birula decomposition of M is a stratification which is locally of product form (work of Jakub Koncki).

Flag varieties

The most studied varieties in our theory are the (generalized) complete flag varieties. The study initiated by

- Aluffi–Mihalcea (csm),
- Aluffi-Mihalcea-Schürmann-Su (mC).

Let $M = GL_n/B_n$ or more generally G/B. The Schubert **cells** are B-orbits of the torus-fixed points. The fixed points are indexed by the Weyl group. In the GL_n-case $\mathfrak{W} = \Sigma_n$ is the permutation group :

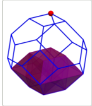
$$\mathfrak{W} \ni \omega \quad \mapsto \quad mC(X_\omega, M) \in K_{\mathbb{T}}(M)$$

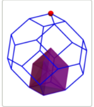
In the equivariant case $mC(X_\omega, M)$ is determined by the set of restrictions to the fixed points $mC(X_\omega, M)_\tau$

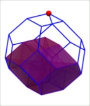
$$mC(X_\omega, M)_{|M^{\mathbb{T}}} \in \bigoplus_{\tau \in \mathfrak{W}} \mathbb{Z}[t_1^\pm, t_2^\pm, \dots, t_n^\pm, y]$$

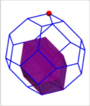
From the equivariant characterization of mC-classes it follows that they coincide with *the trigonometric weight functions defined by Rimányi-Tarasov-Varchenko.*

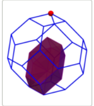
Newton polytopes inclusion at the 0-dimensional cell

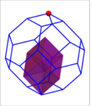

pt4321class1234.png

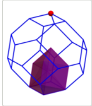

pt4321class1243.png

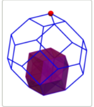

pt4321class1324.png

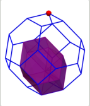

pt4321class1342.png

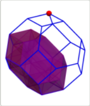

pt4321class1423.png

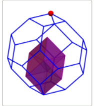

pt4321class1432.png

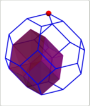

pt4321class2134.png

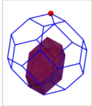

pt4321class2143.png

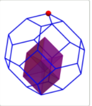

pt4321class2314.png

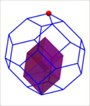

pt4321class2341.png

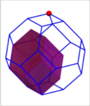

pt4321class2413.png

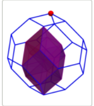

pt4321class2431.png

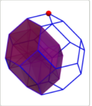

pt4321class3124.png

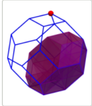

pt4321class3142.png

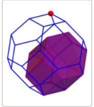

pt4321class3214.png

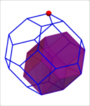

pt4321class3241.png

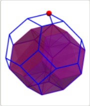

pt4321class3412.png

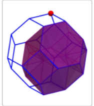

pt4321class3421.png

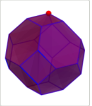

pt4321class4123.png


pt4321class4132.png


pt4321class4213.png


pt4321class4231.png


pt4321class4312.png


pt4321class4321.png

Hecke algebra

Theorem ([Aluffi-Mihalcea-Schürmann-Su, 2019])

There are explicit operators

$$\beta_i \in End\Big(\bigoplus_{\tau\in\mathfrak{W}} \mathbb{Z}[t_1^{\pm}, t_2^{\pm}, \dots, t_n^{\pm}, y]\Big)$$

such that

$$mC(X_{\omega s_i}, M) = \beta_i(mC(X_\omega, M))$$

if $\ell(\omega s_i) > \ell(\omega)$ *and* β_i*'s satisfy the braid relations and Hecke relations*

$$(\beta_i + 1)(\beta_i + y) = 0$$

The operators β_i are versions of versions of Lusztig operator in equivariant K-theory.

Hecke inductive formula allows to compute $\mathrm{mC}(X_\omega)$ effectively.

Expansion in Schubert classes

The mC classes can be written in the basis consisting of the K-theoretic fundamental classes of Schubert cells. Already the *non-equivariant* classes have very particular coefficients. For GL_4:

$$
\begin{aligned}
\mathrm{mC}(X_{4321}) = \; & (y+1)^6\,[\mathbf{4321}] \\
& -(y+1)^5(2y+1)\,[\mathbf{3421}] \\
& -(y+1)^5(2y+1)\,[\mathbf{4231}] \\
& -(y+1)^5(2y+1)\,[\mathbf{4312}] \\
& +(y+1)^4(5y^2+4y+1)\,[\mathbf{2431}] \\
& \quad \vdots \\
& +(y+1)^2(23y^4+35y^3+22y^2+7y+1)\,[\mathbf{4},\mathbf{1},\mathbf{3},\mathbf{2}] \\
& +(y+1)^2(24y^4+36y^3+21y^2+7y+1)\,[\mathbf{4},\mathbf{2},\mathbf{1},\mathbf{3}] \\
& -(y+1)(44y^5+85y^4+66y^3+29y^2+8y+1)\,[\mathbf{3},\mathbf{4},\mathbf{2},\mathbf{1}] \\
& -(y+1)(49y^5+91y^4+69y^3+30y^2+8y+1)\,[\mathbf{1324}] \\
& -(y+1)(44y^5+85y^4+66y^3+29y^2+8y+1)\,[\mathbf{2134}] \\
& +(64y^6+163y^5+169y^4+98y^3+37y^2+9y+1)\,[\mathbf{1234}]
\end{aligned}
$$

Expansion in Schubert classes

- The coefficient of [**12345**] in $\mathrm{mC}(X_{54321})$ is equal to

$$1 + 14y + 92y^2 + 377y^3 + 1120y^4 + 2630y^5 + 4972y^6 + 7148y^7 + 7024y^8 + 4063y^9 + 1024y^{10}.$$

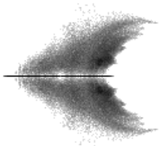
- The coefficient of $[\mathbf{6},\mathbf{5},\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{1}]$ in $\mathrm{mC}[1,2,3,4,5,6]$ is equal to

$$1 + 20y + 191y^2 + 1159y^3 + 4997y^4 + 16383y^5 + 43300y^6 + 97510y^7 + 195761y^8 + 355455y^9 + 563851y^{10} + 727301y^{11} + 702113y^{12} + 463400y^{13} + 183781y^{14} + 32768y^{15}$$

The coefficients are not only nonnegative (up to a prescribed sign), but also they are log-concave.

$$\sum a_i y^i \text{ is log-concave} \Leftrightarrow \forall_i \ a_{i-1}a_{i+1} < a_i^2$$

This statement is checked for GL_n, $n \leq 6$.



Addendum: Beyond K-theory:

	Euler	Chern class	RR image	genus
$H^*(-)$	x	$x+u$	$\underset{u=1}{1+x}$	$\chi(-)$
$K(-)[y]$	$1-\frac{1}{x}$	$1-\frac{1}{ux}$	$\underset{u=-y^{-1}}{x\frac{1+ye^{-x}}{1-e^{-x}}}$	$\chi_y(-)$
Elliptic	$\theta(x)$	$\theta(x+u)$	$\underset{u=-z}{x\frac{\theta(x-z)}{\theta(x)}}$	elliptic genus

RR means Riemann-Roch transformation to $H^*(-)$.

Here $\theta=\theta_\tau$ is the Jacobi theta function $\tau\in\mathbb{H}_+$

$$\theta_\tau(x)=\mathrm{const}\cdot(e^{x/2}-e^{-x/2})\prod_{\ell=1}^{\infty}\left(1-q^\ell e^x\right)\left(1-q^\ell e^{-x}\right), \qquad q=e^{2\pi i\tau}\,.$$

The elliptic characteristic classes (already defined by Hirzebruch, Ochanine, Krichever, Höhn, . . .) do **not** admit an extension for arbitrary singular spaces.
There is no motivic extension.

Elliptic classes

- Borisov-Libgober defined elliptic classes in cohomology. It is assumed that varieties have mild singularities ,,Kawamata log-canonical singularities" (KLC).
- The construction extends to pairs (X, D), where $K_X + D$ is $\mathbb{Q}$-Gorenstein divisor, and the pair is KLC.
- It is forbidden to take $Ell(X, D)$ when the coefficients of D are equal to 1.
- It is not allowed to study $Ell(\bar{X}_\omega, \partial X_\omega)$ but

$$E(X_\omega, \lambda) := Ell(\bar{X}_\omega, \partial X_\omega - \mathcal{L}_\lambda)$$

  for $\lambda \in \mathfrak{t}^*_{\mathbb{Q}}$ makes sense. Here $\mathcal{L}_\lambda$ is the line bundle associated to a weight λ. We assume that λ is strictly dominant.
- The function $\lambda \mapsto E(X_\omega, \lambda)$ extends to a meromorphic function on $\mathfrak{t}^*$.
- For $\lambda = 0$ there is a pole.

Elliptic classes of X_ω cannot be defined, but their deformation can.

Elliptic classes

The classes $E(X_\omega,\lambda)=\{E(X_\omega,\lambda)_\sigma\}_{\sigma\in\mathfrak{W}}$ can be treated as a meromorphic functions on $\mathfrak{t}^*\times\mathbb{R}$. The additional parameter is responsible for the scalar $\mathbb{C}^*$-action.

Theorem ([Rimányi-AW])

There are operators

$$\beta_i \in End\left(\bigoplus\nolimits_{\mathfrak{W}} Mero\big(\mathfrak{t}^*\times\mathbb{R}\,,\;\hat{H}^*_{\mathbb{T}}(pt)[[q]]\big)\right)$$

(given by an explicit formula) such that

$$E(X_{\omega s_i},\lambda)=\beta_i(E(X_\omega,s_i(\lambda)))$$

if $\ell(\omega s_i)>\ell(\omega)$ and β_i's satisfy the braid relations and the square

$$(\beta_i\circ s_i^\lambda)^2=\kappa(\lambda)\,id$$

is the multiplication by the function

$$\kappa(\lambda)=const(q)\;\frac{\theta(z-\lambda)\theta(z+\lambda)}{\theta(z)^2\,\theta(\lambda)^2}\,.$$

Final remark: R-matrix relations

- Elliptic classes for the generalized partial flag varieties G/P.
- The Schubert cells are indexed by $\mathfrak{W}/\mathfrak{W}_P$.

Theorem (Shrawan Kumar-Rimányi-AW)

There are operetors R_i acting on $\hat{H}^*_{\mathbb{T}'}(G/P)[[q]]$ such that

$$R_i\big(E(X_{[\omega]},\lambda\big) = s_i^t E(X_{[s_i\omega]},\lambda\big)\,.$$

where s_i^t acts on the equivariant variables.

Thank you